NWP Equations

(Adapted from UCAR/COMET Online Modules)

Certain physical laws of motion and conservation of energy (for example, Newton's Second Law of Motion and the First Law of Thermodynamics) govern the evolution of the atmosphere. These laws can be converted into a series of mathematical equations that make up the core of what we call numerical weather prediction.

Vilhelm Bjerknes first recognized that numerical weather prediction was possible in principle in 1904. He proposed that weather prediction could be seen as essentially an initial value problem in mathematics: since equations govern how meteorological variables change with time, if we know the initial condition of the atmosphere, we can solve the equations to obtain new values of those variables at a later time (i.e., make a forecast).

To mathematically represent an NWP model in its simplest form, we write

$$\frac{\Delta A}{\Delta t} = F(A)$$

where

- ΔA equals the change in a forecast variable at a particular point in space
- Δt equals the change in time (how far into the future we are forecasting)
- F(A) represents terms that can cause changes in the value of A

The equation can be expressed in words as

The change in forecast variable A during the time period Δt is equal to the cumulative effects of all processes that force A to change.

In NWP, future values of meteorological variables are solved for by finding their initial values and then adding the physical forcing that acts on the variables over the time period of the forecast. This is stated as

$$A^{forecast} = A^{initial} + F(A)\,\Delta t$$

where F(A) stands for the combination of all of the kinds of forcing that can occur.

This stepwise process represents the configuration of the prediction equations used in NWP. The specific forecast equations used in NWP models are called the **primitive equations** (not because they are crude or simplistic, but because they describe the fundamental processes that occur in the atmosphere). These equations govern the motion and thermodynamic changes that occur in the atmosphere and are derived from the complete conservation laws of momentum, mass, energy, and moisture.

The way in which primitive equations are derived from their complete theoretical form and converted to computer codes can contribute to forecast errors in NWP models.

- The model forecast equations are simplified versions of the actual physical laws governing atmospheric processes, especially cloud processes, land-atmosphere exchanges, and radiation. The physical and dynamic approximations in these equations limit the phenomena that can be predicted.
- Due to their complexity, the primitive equations must be solved numerically using algebraic approximations, rather than calculating complete analytic solutions. These numerical approximations introduce error even when the forecast equations completely describe the phenomenon of interest and even if the initial state were perfectly represented.
- Computer translations of the model forecast equations cannot contain all details at all resolutions. Therefore, some information about atmospheric fields will be missing or misrepresented in the model even if perfect observations were available and the initial state of the atmosphere were known exactly.
- Grid point and spectral methods are techniques for representing information about atmospheric variables in the model and solving the set of forecast equations. Each technique introduces different types of errors.

The Primitive Equations

The five equations shown below govern changes in the motion and thermodynamics of the atmosphere and are derived from the complete set of conservation laws of momentum, mass, energy, and moisture.

Spectral and grid point models use the same set of governing equations to describe changes that occur at individual locations within the forecast model. The equations shown here represent a simplified set of the actual equations used in NWP models. They are written in the Eulerian framework, in which values and

Wind Forecast Equations

$$\text{1a.}\quad \frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \omega\frac{\partial u}{\partial p} + fv - g\frac{\partial z}{\partial x} + F_x$$

$$\text{1b.}\quad \frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - \omega\frac{\partial v}{\partial p} - fu - g\frac{\partial z}{\partial y} + F_y$$

Continuity Equation

$$\text{2.}\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

Temperature Forecast Equation

$$\text{3.}\quad \frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} - \omega\left(\frac{\partial T}{\partial p} - \frac{RT}{c_p p}\right) + \frac{H}{c_p}$$

Moisture Forecast Equation

$$\text{4.}\quad \frac{\partial q}{\partial t} = -u\frac{\partial q}{\partial x} - v\frac{\partial q}{\partial y} - \omega\frac{\partial q}{\partial p} + E - P$$

Hydrostatic Equation

$$\text{5.}\quad \frac{\partial z}{\partial p} = -\frac{RT}{pg}$$

their derivatives (changes to the variable over time, for example, $\frac{\partial T}{\partial t}$, or space $\frac{\partial T}{\partial x}$) are evaluated at fixed locations on the earth. They are also written in pressure (x-y-p) coordinates and contain all of the essential physics and dynamics needed for NWP models, except that terms considering the earth's curvature have been left out and physical processes, such as friction and diabatic heating, are represented as one term rather than many. Because pressure coordinates are used in this example, the pressure gradient force (PGF) is expressed as a gradient in height (z), and "horizontal" flow is assumed to occur on isobaric surfaces. (Although we have used isobaric coordinates in these simplified equations, most NWP models use other vertical coordinate systems to improve accuracy and simplify computations.)

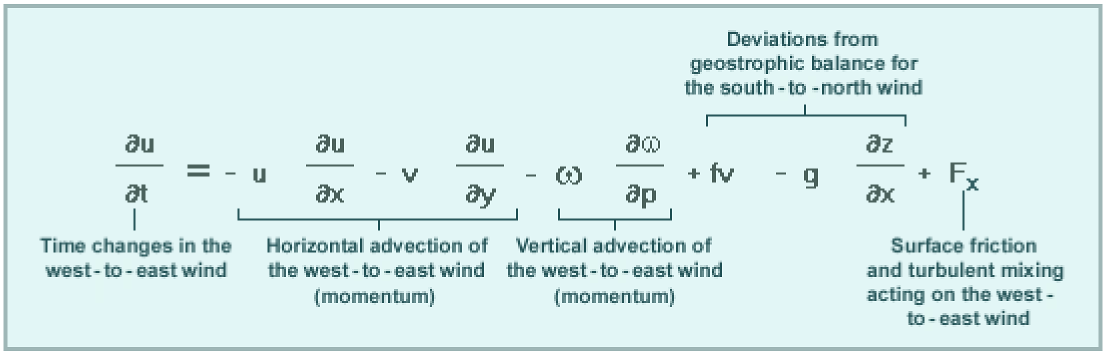
For forecasting purposes, this set of equations is considered to be closed and complete (meaning that we can forecast values of all terms by solving each of the equations in the proper sequence) since

- All equations use the same basic forecast variables (u, v, ω, T, q, and z)

- The terms F_x, F_y, H, E, and P can also be described in terms of the six basic forecast variables
- We can specify initial conditions over the domain of the model
- We can obtain suitable boundary conditions for all forecast variables at the boundaries of the model

Wind Forecast Equations

West-to-East Component



$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \omega\frac{\partial \omega}{\partial p} + fv - g\frac{\partial z}{\partial x} + F_x$$

This equation determines time changes in the west-to-east component of the wind caused by

- Horizontal advection of west-to-east wind
- Vertical advection of west-to-east wind
- Deviations from the geostrophic balance of the south-to-north wind component. Imbalances between the west-to-east pressure gradient force and the Coriolis force acting on the south-to-north wind will change the west-to-east wind
- Other physical processes, such as surface friction and turbulent mixing acting on the west-to-east wind. The models also include empirical approximations to try to account for atmospheric processes that cannot be forecast directly, although some of the effects are indirect. For example, radiation and convection are applied only to the temperature and moisture equations and are not included explicitly in the wind forecast equations. However, the changes in temperature at one time will cause changes in the pressure gradient, which in turn will affect the wind at a later time

Note that the two wind components are interrelated -- each is affected by geostrophic imbalances in the other.

South-to-North Component

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - \omega\frac{\partial v}{\partial p} - fu - g\frac{\partial z}{\partial y} + F_y$$

Deviation from geostrophic balance for the west - to - east wind ($-fu - g\frac{\partial z}{\partial y}$)

TIme changes in the south - to - north wind ($\frac{\partial v}{\partial t}$)

Horizontal advection of the south - to - north wind (momentum) ($-u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y}$)

Vertical advection of the south - to - north wind (momentum) ($-\omega\frac{\partial v}{\partial p}$)

Surface friction and turbulent mixing acting on the south - to - north wind (F_y)

This equation determines time changes in the south-to-north component of the wind caused by

- Horizontal advection of south-to-north wind
- Vertical advection of south-to-north wind
- Deviations from geostrophic balance of the west-to-east wind component. Imbalances between the west-to-east pressure gradient force and the Coriolis force acting on the south-to-north wind will change the west-to-east wind
- Other physical processes, such as surface friction and turbulent mixing acting on the south-to-north wind. The models also include empirical approximations to try to account for atmospheric processes that cannot be forecast directly, although some of the effects are indirect. For example, radiation and convection are applied only to the temperature and moisture equations and are not included explicitly in the wind forecast equations. However, the changes in temperature at one time will cause changes in the pressure gradient, which in turn will affect the wind at a later time

Note that the two wind components are interrelated -- each is affected by geostrophic imbalances in the other.

Additional Information: Coriolis Force

To illustrate a conceptual example of the effects of the Coriolis force, the wind (momentum) equations are simplified by assuming that advection and frictional effects are equal to zero for an atmosphere initially at rest. Using this assumption, the equations reduce to the following form.

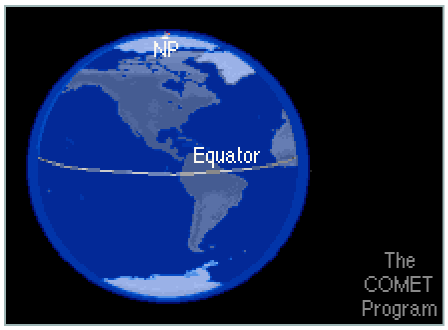
1) $$\frac{\Delta u}{\Delta t} = fv - g\frac{\Delta z}{\Delta x}$$

2) $$\frac{\Delta v}{\Delta t} = -fu - g\frac{\Delta z}{\Delta y}$$

- $\Delta u/\Delta t$ and $\Delta v/\Delta t$ are the rates of changes in the u and v components of the wind during the time step Δt

- fv and fu represent the Coriolis effect
- ($g\,\Delta z/\Delta x$ and $g\,\Delta z/\Delta y$) represent the pressure gradient accelerations in the x and y directions (the change in z over distance)

Initially, we assume that the earth is not rotating and that there is no solar radiation. With the atmosphere at complete rest, each of the three terms in the momentum equation would equal zero. We will first examine the effects on air parcel motion by introducing solar radiation, then the earth's rotation. The following discussion applies to the Northern Hemisphere.



As incoming solar radiation comes into play, the equator heats up more than over the poles, creating a south-to-north temperature gradient with temperature decreasing to the north, as illustrated in the animation. Because warm air has greater thickness than colder, denser air, the upper-level pressure surfaces become higher over the equator than over the poles and a north-to-south pressure gradient develops. In the equation, the north-to-south gradient term becomes

$$-g(\Delta z/\Delta y) > 0$$

When this term is positive, a northward acceleration is created with air essentially moving down the pressure gradient from south to north (it moves downhill). This means that the v component of the wind is also positive and a positive v component is physically manifest as a southerly wind.

$$\Delta v/\Delta t > 0$$

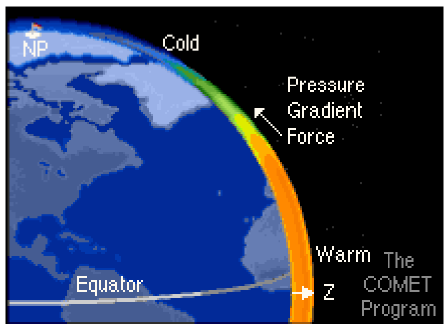
We have not yet considered the Coriolis effects. In the real atmosphere, the earth is rotating and the Coriolis force is not zero except at the equator. Recall that the Coriolis effect results in a deflection of the air to the right in the Northern Hemisphere ($fv > 0$).

If $fv > 0$, then $\Delta u/\Delta t > 0$. This indicates an eastward acceleration since positive u motion indicates a westerly component to the wind.

Now that the wind has an eastward component, the Coriolis term (fu) in equation 2 must also be considered. In this case, since $u > 0$, then $-fu < 0$, reducing the northward component of the wind.

$$\Delta v/\Delta t < 0$$

This negative acceleration reduces the southerly wind component and eventually, over several hours, the wind becomes northerly ($v < 0$), deflecting the air parcel toward the south. This interaction of the pressure gradient forces and Coriolis effects results in an oscillation, as illustrated below.



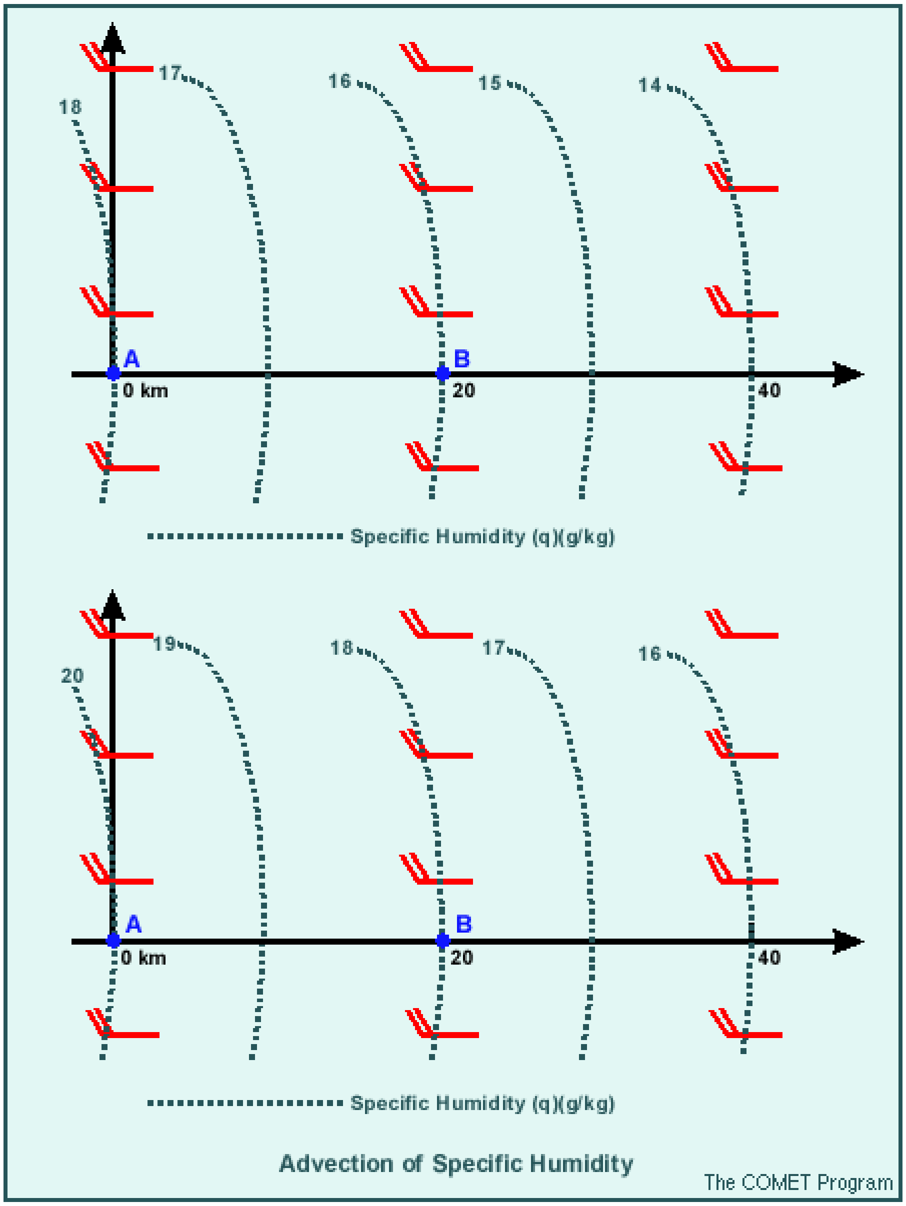
It is important to note the following.

- In this example, the air parcel undergoes inertial oscillations in the absence of other pressure gradient forces. In nature, however, these oscillations are usually quickly damped and a balanced flow regime is established.
- The atmospheric response to a pressure gradient force has been presented as a series of sequential events. In the real atmosphere, these responses occur simultaneously as the wind and pressure fields continuously adjust to each other.

Additional Information: Advection

The graphic depicts an idealized situation to explain how advection of a quantity by the wind works.

The advection term determines how the value of a variable at a fixed location will change due only to the motion of a feature being carried by the wind. For example, at time T = 0, the top panel shows contours of increasing specific humidity associated with a moist air mass near point A and drier conditions at point B. The wind velocity is constant at 20 knots from west to east. As the wind moves this pattern toward the right, over time, the shape of the moisture pattern remains the same but the pattern shifts. The specific humidity value at point B increases as the moist air mass moves over the area. The amount of change is directly proportional to the strength of the wind moving the moisture pattern and the strength and location of the spatial variability (gradients) in the moisture field.



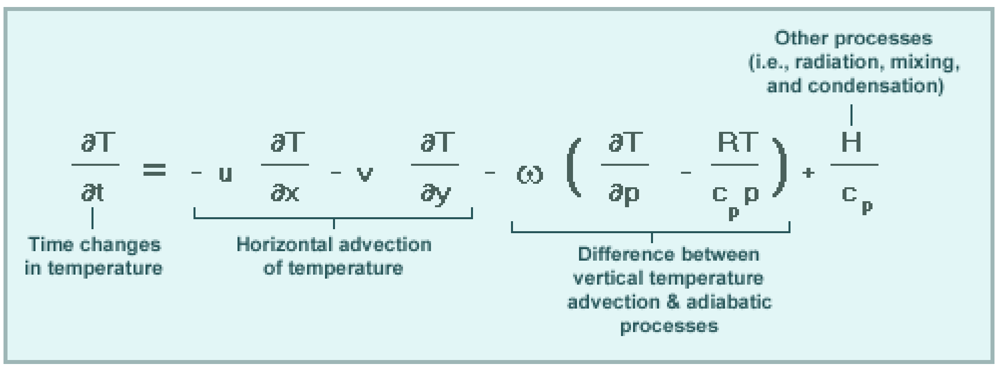
Continuity Equation

$$\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}_{\text{Horizontal Divergence}} + \underbrace{\frac{\partial \omega}{\partial p}}_{\text{Vertical Divergence}} = 0$$

In this example, the continuity equation is calculated diagnostically from the horizontal wind fields without considering buoyancy effects. Horizontal divergence is determined from the spatial variations in both of the horizontal wind components. The divergence is then related to the change in vertical motion from the bottom to the top of a layer within the model. Areas of horizontal convergence must coincide either with areas where rising motion increases with height or where sinking motion weakens with height.

The continuity equation is used to calculate vertical motion in **hydrostatic** models. **Non-hydrostatic** models, on the other hand, do not use the continuity equation directly to calculate vertical motion. Rather, they use a combination of horizontal divergence and buoyancy to determine both vertical motions and vertical accelerations.

Temperature Forecast Equation



$$\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y} - \omega\left(\frac{\partial T}{\partial p} - \frac{RT}{c_p p}\right) + \frac{H}{c_p}$$

Time changes in the temperature are related to

- Horizontal advection of temperature by both wind components
- The difference between vertical advection of temperature and cooling or warming caused by expansion or compression of rising or sinking air. This component of the temperature change is proportional to the intensity of vertical motion and the difference between the forecast lapse rate and the dry adiabatic lapse rate
- The effects of all other processes, notably radiation, mixing, and condensation, including the effects of convection

Note the importance of the vertical velocity determined from the wind forecast and continuity equations. This equation is also dependent upon the moisture forecast equation because of the role of moisture in the amount of condensational heating and cooling and in the triggering of convection, which also contributes to condensational heating and cooling.

Moisture Forecast Equation

$$\underbrace{\frac{\partial q}{\partial t}}_{\text{Time changes in moisture}} = \underbrace{- u \frac{\partial q}{\partial x} - v \frac{\partial q}{\partial y}}_{\text{Horizontal advection of moisture}} \underbrace{- \omega \frac{\partial q}{\partial p}}_{\text{Vertical advection of moisture}} + \overbrace{E}^{\text{Evaporation and sublimation}} - \underbrace{P}_{\text{Condensation (Precipitation)}}$$

Time changes in moisture are related to

- Horizontal advection of moisture
- Vertical advection of moisture
- Evaporation of liquid water or sublimation of ice crystals
- Condensation (precipitation). Models have many complicated formulations for estimating condensation and subsequent precipitation. Note that conservation of moisture means that precipitation predicted by the model reduces the amount of moisture in the model atmosphere. Thus, when a model incorrectly forecasts precipitation, the amount of moisture downstream is affected

Note the importance of the vertical velocity determined from the wind forecast and continuity equations. There is also interdependency between the forecast temperature equation and the amount of evaporation that can be expected from the earth's surface

Hydrostatic or Vertical Momentum Equation

$$\overbrace{\frac{\partial z}{\partial p}}^{\text{Difference in height between upper \& lower isobaric sufaces}} = - \frac{\overbrace{RT}^{}}{\underbrace{pg}_{\text{Mean temperature within a layer}}}$$

The hydrostatic equation preserves stability within the forecast model and is used to calculate the height field necessary for determining geostrophic balance in the wind forecast equations. This diagnostic equation links the mean temperature in a layer of the model to the difference in height between the upper and lower isobaric surfaces serving as the top and bottom of the layer. Updated temperatures obtained from the temperature forecast equation are used here to calculate heights, which are then used in the wind forecast equations.