On Formation and Intensification of Tropical Cyclones Through Latent Heat Release by Cumulus Convection

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ABSTRACT

The effect on large scale motions of latent heat release by deep cumulus convection in a conditionally unstable atmosphere is investigated and a method devised to include this effect directly in the equations for large scale flow. This method is then applied to the hurricane formation problem by incorporating it into time-dependent, circular symmetric dynamic hurricane models, either in gradient-wind balance or unbalanced.

Numerical integrations of a two-level approximation of the balanced model have been carried out for two different formulations of the problem (including or not including a frictional radial flow), both starting from a hypothetical initial state characterized by a weak barotropic circular vortex with a maximum tangential velocity of 10 m sec^{-1} at a distance of 141.2 km from the center. The results obtained without frictional radial flow showed slow intensification of the tangential flow, to about 25 m sec^{-1}, and establishment of a strong radial temperature gradient in the upper troposphere, from sixteen to twenty-four hours after the initial time, after which a steady state ensued. The radial flow obtained from this model remained less than 2 m sec^{-1}. On the other hand, the results obtained with a superimposed frictional radial flow either decayed after reaching a moderate tangential velocity, or developed very rapidly after attaining higher velocity, and did not approach any steady state. The results further show that while the two-level approximation of the balanced model is able to reveal many important aspects of the development problem, it is not able to describe the further development associated with the upper level temperature gradient.

1. Introduction and summary

Broadly speaking, the life history of a typical hurricane or typhoon may be divided into three stages, namely, an infant, cold-core wave-perturbation stage, a more or less rapidly developing transitional stage, and a strong, mature, warm-core vortex stage.

Concerning the first stage, observational studies show that hurricanes and typhoons always originate from some pre-existing large-scale perturbation, such as equatorial and easterly waves; therefore the infant stage may be identified with these wave perturbations. The exact nature of these wave perturbations in the easterlies is still not well understood. Judging from the fact that their regions of ascending motion are colder than their descending regions, it seems likely that they are initiated by barotropic instability of the mean wind.

When the easterly or equatorial waves move from the eastern part of the ocean to the western part, they often develop into vortices, especially in the middle troposphere, but their cores still remain relatively cold (see Palmer, 1952) and deepening of surface pressure seldom takes place. This development from wave perturbation to vortex motion can be due either to the sustained growth of the wave motion, or to a new instability associated with a changed environment, such as the increase of the easterly current with height that exists in the western parts of the Pacific and Atlantic Oceans.

However, under more favorable conditions, some of these cold-core, high-level vortices transform into deep warm-core vortices, with the change of temperature usually taking place first in upper levels and then spreading downward (see Yanai, 1961), indicating that the change is due primarily to the release of latent heat of condensation. After this change of the thermal structure has taken place, it becomes a baroclinic convective system instead of a barotropic system, the surface pressure deepens rapidly, and the tangential wind intensifies until another equilibrium state is reached.

In the past, many linear theories (Haque, 1952; Syōno, 1953, Lilly, 1960; and many others) and nonlinear theories (Kasahara, 1961) on tropical cyclone formation have been advanced on the premise that the tropical storms develop from the limiting axially symmetric perturbations in a conditionally or absolutely unstable atmosphere. The difficulties of such theories have been pointed out by the author in two previous papers (Kuo, 1960, 1961) wherein the results obtained from linear analysis show that under such conditions, the energy tends to go into establishing strong, narrow, ring-shaped convection cells instead of cyclone-scale perturbations. Another aspect of this kind of development is the creation of a too intense meridional circulation. After this difficulty was realized several remedies were suggested: the inclusion of various types of diffusion processes which tend to shift the preferred scale of convection from that of a cloud to that of a cyclone (e.g., Lilly, 1960; Kasahara, 1961; Kuo, 1961), and the introduction of area averages to eliminate small-scale convection (Yanai, 1961).

Rosenthal's (1964) recent numerical work is along this line and is most interesting. By taking a large horizontal grid-mesh and thereby introducing a large damping effect on the small-scale circulation, he was able to carry his computations up to ten (10) hours of physical time. He obtained strong vortices surpassing hurricane strength from a relatively weak initial vortex. However, the meridional circulations he obtained are still rather too strong, the large radial temperature gradients are concentrated in the lower part instead of the upper part of the troposphere, and the rapid development does not seem to stop. Therefore many of the difficulties associated with this model remain to be removed, and more investigation in this direction is needed.

Partly because of the realization of the above mentioned difficulties of the mean-vertical motion model, and partly because of Riehl and Malkus' (1960) definitive observational results that indicate the decisive role of the deep cumulus towers in releasing latent heat, many dynamic meteorologists who are engaged in the hurricane intensification problem tried to find a different approach. For example, a different mechanism was proposed by the writer at a seminar given at Lamont Observatory of Columbia University in January 1962, in which he attributed the release of latent heat to the deep convective cloud-bands instead of to the axially symmetric mean field. The present work is a formalization of this concept, and a brief summary of this work (Kuo, 1963b) has been reported at the Third Technical Conference on Hurricane, and Tropical Meteorology held at Mexico City, in June 1963, where similar arguments were also reported. Among these we have Charney *et al.* (1963), Kasahara (1963), and Ooyama (1963), while the work of Syōno (1963) was reported more recently at the XIIIth Assembly of the IUGG. Since the specific models proposed by these authors differ greatly from ours, especially in regard to the specification of the liberation of latent heat and the role of surface friction, we shall present our work here in some detail.

Concerning the effect of the latent heat on the large scale circulation, our basic assumption is that, in the region of conditionally unstable stratification, the release of latent heat of condensation is accomplished by the vertical motion associated with deep cumulus convection, and not through the vertical motion of the mean wind field. However, because of the control of the water vapor supply by the low level mean flow field, these deep cumulus convections serve collectively as a heat source for the mean flow field, in addition to their function as agents of diffusion. Thus our first task is to include properly this heating effect on the large scale motion.

We have two main objectives in this paper: First we shall show that the statistical effect of the convective motions can be included without referring to their details by using a certain averaging process, and then we shall derive the formulas that express the latent heat released by the deep cumulus purely in terms of parameters of large scale quantities. This is accomplished through the application of a very simple cloud model, which is essentially the ejection of mean surface air from a source point. Secondly, we shall apply this method of including the latent heat released by deep cumulus convections to the investigation of the hurricane development problem, by incorporating this heat input in a simplified version of a time-dependent dynamic model and integrating it numerically.

The physical reason for the necessity and feasibility of the separation of the energy-releasing motion field from the mean motion field in the case of conditionally unstable stratification and the exact manner in which the latent heat is to be included in the mean equations are discussed in detail in Section 3. It is shown that under the assumption that deep cumulus convective motions bring the moist surface air directly to high levels, the time changes of temperature and mixing ratio can be determined from the horizontal advection of humidity and the vertical temperature and humidity distributions. In Section 4 the flux of energies and momentum through the lower and upper boundaries and their relations to the kinetic and thermal energy integrals are discussed, while in Section 5 the pertinent equations for the development problem are derived from a two-level approximation of an axially symmetric balanced model. The purpose of using this simple model is to elucidate the most important features of the development so as to facilitate further works with the inclusion of this mode of heat input in more adequate dynamic models.

Actual computations with the two-level balanced model have been carried out from six different formulations of the approximation, both with and without an Ekman boundary layer, all starting from the same hypothetical initial state represented by a weak, barotropic circular vortex in gradient-wind balance. The integrations of the group I experiments, which employ the model without a separate Ekman layer, have been continued to 36 hours, while that of the group II experiments employing a model with superimposed frictional flow have been carried out to somewhat shorter periods. The results of these integrations are presented in Section 6. It is shown that for the group I experiments the maximum low level tangential wind increased from the original 10 m sec^{-1} to more than 25 m sec^{-1} and its position shifted from the original 142 km from the center to between 60 km to 90 km from the center, while the radial wind remained small and decreased in intensity in the later parts of the integrations, showing that relatively steady states were being approached. Further, the warming of the central core is more pronounced in upper levels, as is expected from the liberation of latent heat by the deep cumulus elements. On the other hand, integrations with the frictional inflow model either resulted in the development of a large inner reverse cell and thereby lead to decay, or in unlimited and unmanageable development. This breakdown of the system is particularly rapid when the frictional flow is determined by the customary requirement of balance between surface friction and radial advection of absolute angular momentum of the boundary layer.

Because of the lack of a heat sink in the symmetric model and lack of a proper mechanism to utilize the potential energy in the upper layer in the two-level approximation of the balanced model, the mean temperature of the system rises continuously. It has also been found that vertical diffusion has a significant effect on the temperature variations, especially near the surface, and it therefore needs a more adequate formulation.

2. Basic equations

Since the main features of well developed hurricanes are nearly axially symmetric, it is convenient to use a cylindrical coordinate system (r,λ,z), with the z-axis coinciding with the center of the storm, which will be assumed to be stationary. In order to concentrate our attention on the circularly symmetrical part of the system, and at the same time to make it possible to include some contributions from the non-symmetrical parts of the various quantities, we split every quantity $\mathbf{X}$ into a symmetric part X and a departure X', such that

$$\mathbf{X}=X+X',$$

$$X=\frac{1}{2\pi}\int_0^{2\pi}\mathbf{X}d\lambda, \qquad \int_0^{2\pi}X'd\lambda=0. \tag{2.1}$$

We consider that all the relevant equations have been so averaged that the terms involving λ have been eliminated.

As the pressure force in the vertical is almost exactly balanced by gravity in the large scale systems, complete hydrostatic equilibrium may be assumed. It is then advantageous to use the non-dimensional pressure $\zeta=p/P$ as the vertical coordinate instead of z. The equations of motion in the λ, r, and z-directions, the continuity equation and the thermodynamic equation appropriate to our circularly averaged system may then be written as follows:

$$\frac{du}{dt}+\left(f+\frac{u}{r}\right)v=-J_u, \tag{2.2}$$

$$\frac{dv}{dt}-\left(f+\frac{u}{r}\right)u=-\frac{\partial\phi}{\partial r}-J_v, \tag{2.3}$$

$$0=-\frac{\partial\phi}{\partial\zeta}-R\zeta^{\kappa-1}\theta, \tag{2.4}$$

$$\frac{\partial vr}{\partial r}+\frac{\partial\omega r}{\partial\zeta}=0, \tag{2.5}$$

$$\frac{d\theta}{dt}=\frac{\zeta^{-\kappa}}{c_p}(Q_e+Q_r)-\zeta^{-\kappa}J_r, \tag{2.6}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial r}+\omega\frac{\partial}{\partial\zeta}, \qquad \kappa=R/c_p,$$

u and v are the relative tangential and radial velocity components, $\omega=d\zeta/dt$, ϕ is the geopotential of the isobaric surface, θ is the potential temperature, Q_e and Q_r are the rates of accession of heat to a unit mass by latent heat release and by radiation, and the various J's represent the eddy transports of momentum and heat, given by

$$J_x=\frac{\partial(\overline{v'x'r})}{r\partial r}+\frac{\partial\overline{\omega'x'}}{\partial\zeta}. \tag{2.7}$$

Since the release of latent heat of condensation is the main source of energy for the maintenance and development of tropical storms, another equation for the water vapor budget is needed in order to calculate this supply of energy. This equation may be written as

$$\frac{dq}{dt}=\frac{dm}{dt}-J_q, \tag{2.8}$$

where q is the mixing ratio, defined in terms of the vapor pressure e by

$$q=\frac{0.622e}{p-e}, \tag{2.9}$$

dm/dt is the accession of moisture by evaporation in the volume occupied by the unit mass per unit time, or the subtraction of moisture by condensation when q exceeds its saturation value q_s.

The relation between the rate of accession of latent heat Q_e and the motion-field depends very much on the way the latent heat is released. Broadly speaking, there are two different ways in which the latent heat of condensation can be released, viz: (i) directly through the large-scale vertical motion and (ii) through the small-scale, deep cumulus convection. In this section we shall discuss only the inclusion of Q_e through the first mechanism and its proper place of application, while the inclusion of Q_e through the second mechanism will be discussed in the next section.

For mathematical simplicity, we shall assume that the condensation processes are pseudo-adiabatic, i e., all products of condensation fall out of the system as precipitation, and storage and re-evaporation of liquid water in the air will not be allowed. Then the latent heat released in a unit mass of air during a unit time is $-L(dq_s/dt)$, where q_s is the saturation mixing ratio, and L is the latent heat of condensation. Since the individual change of q_s is produced mainly by cooling in ascent, Q_e may be taken as given by the following expression:

$$Q_e=-L\frac{dq_s}{dt}=-L\omega\frac{\partial q_s}{\partial \zeta}, \quad \text{when } q>q_s \text{ and } \omega<0 \tag{2.10}$$
$$=0 \quad \text{when } q<q_s \text{ or } \omega>0.$$

This formula may be considered as valid in general, provided that the ω-field that actually releases Q_e is used. Note that the substitution of this relation in the thermodynamic equation means that the effective stability factor in the saturated ascending region is given by

$$\frac{1}{\theta_E}\frac{\partial \theta_E}{\partial \zeta}=\frac{1}{\theta}\frac{\partial \theta}{\partial \zeta}+\frac{L}{c_pT}\frac{\partial q_s}{\partial \zeta}, \tag{2.11}$$

where Q_E is the equivalent potential temperature.

Since our main interest is the large-scale circulation, it is customary to take ω in (2.10) as representing the mean vertical velocity field. In that event, the relation (2.10) must be restricted to absolutely stable regions only, whereas in the absolutely or conditionally unstable regions another relation must be obtained, in order to avoid the rapid development of strong, small-scale convection cells in general and narrow circular ring type convection cells in the circular symmetric model, as has been predicted by Kuo's (1960, 1961) linear analysis and demonstrated by Kasahara's (1961) numerical integration of the nonlinear equations.

3. Inclusion in the mean equations of the latent heat liberated by deep cumulus convection

As has been mentioned before, our basic assumption concerning Q_e is that in the conditionally unstable regions it is released by the deep cumulus convections, and not by the mean vertical motion. However, because the supply of water vapor in tropical storms is controlled by the converging low level mean flow, both the intensity and the area coverage of the convective motions increase inward, resulting in an inward increase of the statistical heating effect and a radial temperature gradient at high levels, which tends to drive the large-scale system. Thus, statistically speaking, these deep cumulus convections act collectively as a heat source for the mean motions, besides being agents of diffusion for the mean field. This statistical effect of the cumulus convections becomes more evident when one considers the special orientations of the convection systems created by conditional instability in the tropical storms, which take the form of spiral bands, with their axes nearly parallel to the low level mean flow. Thus these convective motions are primarily confined to the plane perpendicular to the local mean velocity, and therefore they have very little direct dynamic influence on the large scale system. However, they do accomplish a very important thermodynamic function by transporting water vapor and latent and sensible heat upward, and by creating a concentration of heat in the inner part of the system through their special orientation and through the convergence of the mean flow. Thus, from the thermodynamic point of view, these convective motions are an indispensable integral part of the whole system, and their effect must be included in a proper way, either directly by a non-symmetric model capable of describing such motions, or indirectly by eliminating their explicit presence from the equations for the mean quantities but including their net effect on the system in some other way. Our present approach is through the second method; that is, we eliminate the explicit appearance of these convective motions from the mean flow equations by applying an averaging process appropriate to this problem.

For simplicity, let us assume that it is possible to select an area A which is small enough to be considered an elementary area for the specification of the various quantities of the large-scale flow and big enough to contain a large number of convection units. By writing every quantity as the X sum of its average $\bar{X}$ in A and the departure $X'=X-\bar{X}$, which we shall attribute to convection, and integrating the equations of motion over A, we then obtain the equations of mean motion (2.2)–(2.5) which involve Reynolds eddy stress terms, on the assumption that the convective velocity is unrelated to the mean velocity. On the other hand, the integration of the thermodynamic equation yields

$$\begin{aligned}\frac{\partial \theta}{\partial t}+V\cdot\nabla\theta+\omega\frac{\partial \theta}{\partial \zeta}&=\frac{Q_r}{c_p\zeta^\kappa}-\frac{\partial\overline{\omega'\theta_E}}{\partial \zeta}\\&=\frac{1}{c_p\zeta^\kappa}(Q_r+Q_e)\end{aligned} \tag{3.1}$$

which shows that, under the present assumption, the rate of the release of latent heat Q_e is given by $-\partial\omega'\theta_E/\partial\zeta$, and is not directly related to the mean vertical motion ω. Our problem now is to find a way of determining the heating rate Q_e produced by cumulus convection in terms of the large scale fields only, without referring to the convective velocity field ω'. When Q_e is known, the change of θ or T can be obtained from (3.1) or (2.6) as the sum of two parts: the first part is obtained from (3.1) or (2.6) without Q_e, while the second part results from the heating Q_e.

To obtain a formula for the accession of latent heat Q_e and the changes of T and q distributions resulting from deep cumulus convection we need to know the heat and moisture contents of the cloud and the way these quantities are imparted to the environmental air. In order to be definite we make the following simple assumptions concerning such deep cumulus clouds:

i) Cumulus convection always occurs in regions of deep layers of conditionally unstable stratification and mean low level convergence.
ii) Such convective motions bring surface air to all levels up to a great height so that inside the cloud the vertical distributions of temperature and mixing ratio are those of the moist adiabat through the appropriate condensation level.
iii) The base of the cloud is at the condensation level of the surface air and the top extends to the level where the moist adiabat through the condensation level meets the environmental temperature profile, or somewhat higher.
iv) The cumulus clouds exist only momentarily. They dissolve by mixing with the environmental-air at the same level, so that the heat and moisture carried up by the cloud air are imparted to the environmental air.

It is easy to see that the assumptions (ii) and (iii) are equivalent to the assumption that the individual cloud is composed of a chain of particles that have originated from the same level and with the same property, which we shall designate as that of the mean surface air. From a dynamic point of view, such clouds may be considered as the products of similar point sources from which mean surface air is streaming up vertically. In the absence of mixing during the formation stage, the temperature distribution in such a jet of cloud-air must be given by the temperature T_s of the moist-adiabat above the condensation level, as is illustrated in Fig. 1, while the humidity distribution will be given by the saturation mixing ratio $q_s(T_s)$ corresponding to T_s. Thus our model cloud is similar to the deep cumulus towers described by Riehl and Malkus (1961).

Objection may be raised about the lack of entrainment of this simple cloud model during the formation stage. However, since our purpose is to calculate the effective heating produced by the cloud on the mean atmosphere, and since complete mixing is assumed after the creation of the cloud, this lack of entrainment in the cloud itself will not affect our latent heat computation but merely makes the calculation simpler.

The assumption (iv) is based on the fact that, from the point of view of the large-scale dynamics, the life duration of the individual clouds is rather short. They develop rapidly and then dissolve, imparting their sensible heat and water vapor to the mean surroundings. This process may be considered as being accomplished through lateral mixing, which introduces the products of the small-scale convective system into the large-scale system.

Four reasons can be cited for treating the deep cumulus towers as a momentary feature rather than a steady, static feature. First, if the clouds are static and occupy only a small fraction of the sky while all the

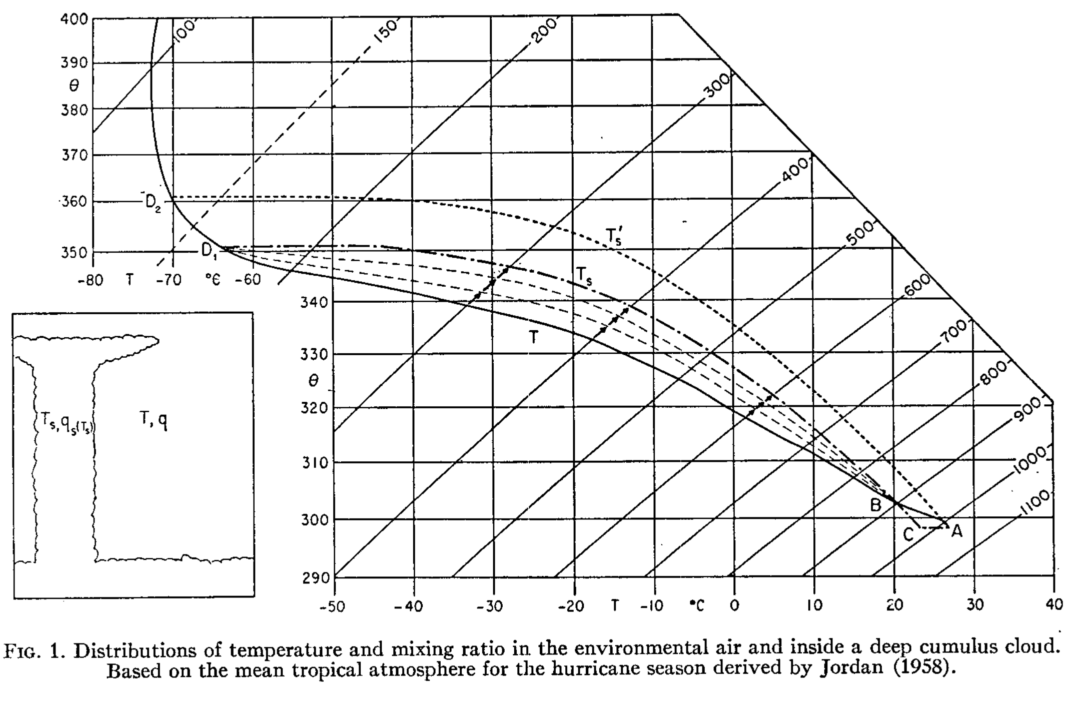

FIG. 1. Distributions of temperature and mixing ratio in the environmental air and inside a deep cumulus cloud. Based on the mean tropical atmosphere for the hurricane season derived by Jordan (1958).

moisture supply is funnelled through them, strong local motion has to be considered. Secondly, the temperature of the mean air cannot be defined except by specifying a way of imparting the heat liberated in the cloud to the mean air, leaving the cloud unaffected. Thirdly, if the moisture supply goes into the existing cumulus tower, the temperature of the cloud cannot be altered even though all the vapor supply will be condensed. Fourthly, if this liberated heat is imagined to be imparted to the mean air without introducing mixing, and without taking the cloud temperature as a reference, then heating will go on indefinitely without limit, as long as there is supply of moisture, since then the humidity of the mean air will not be changed and therefore the mean air will remain permanently unsaturated, even in the region of mean ascending motion. It seems that this last mentioned effect will remain a serious drawback of such a scheme while the other aspects may be eliminated through imaginary mixing consideration without affecting the existing cloud, if one wishes to make use of such imaginary devices.

These considerations indicate that our mixing assumption is both physically realistic and at the same time useful. Its introduction not only relieves us from the necessity of making use of averaging devices which are difficult to comprehend, and a situation with unlimited heating, but also enables us to compute the amount and the vertical distribution of the latent heat input. Thus, if at any moment the deep cumulus clouds cover a fraction $b(z)$ of the area of the sky, and the temperature and mixing ratio of the air outside the clouds at the level z are $T(z)$ and $q(z)$, while that of the clouds are $T_s(z)$ and $q_s(T_s)$, respectively, then the temperature $T^*(z)$ and mixing ratio $q^*(z)$ of the mean air at the level z after mixing will be given by

$$T^*(z)=[1-b(z)]T(z)+b(z)T_s(z) \qquad (3.2)$$

$$=T(z)+b(z)[T_s(z)-T(z)],$$

$$q^*(z)=q(z)+b(z)[q_s(T_s)-q(z)]. \qquad (3.3)$$

We mention that the amount of heat added to the mean air through the cloud is given by $c_pb(T_s-T)$, which represents the change of the internal plus potential energy, a quantity we shall make use of later in calculating the amount of moisture needed to make a cloud.

In this work it will be assumed that all the condensed water falls out of the cloud as precipitation and so no re-evaporation takes place on account of mixing; otherwise the temperature T^* will be lowered while the q^*-value will be increased.

Thus once we have accepted the assumption (iv) of complete lateral mixing of the cloud-air and the surrounding-air after the formation of these deep cumulus clouds, then the amounts of sensible heat and water vapor supplied by such clouds can be determined if we know the percentage area $b(z)$ and temperature and humidity differences $T_s(z)-T(z)$ and $q_s(T_s)-q(z)$. Further, if we know the average rate of production of cumulus cloud over a unit area, we can also determine the heating rate $Q_e(z)$ and the rate of accession of moisture through cumulus convection at any level. Therefore, the determinations of the heating rate Q_e and the rate of accession of moisture at any point reduce to the determinations of:

(a) the temperature difference between the cloud air and environmental air, and
(b) the rate of production of the cloud air.

In what follows, we shall at first determine the temperature distribution T_s inside the cloud and then determine the rate of production of the model cloud air. Thus our computations of Q_e and the T and q distributions begin with the determinations of the temperature and humidity distributions inside the cumulus cloud.

Since the temperature T_s inside the cloud is that of the representative moist adiabat, its vertical variation can be obtained from Clapeyron's equation and is given by

$$\frac{dT_s}{d\zeta}=\frac{ART_s}{c_p\zeta}\,\frac{1+\dfrac{0.622Le_s}{ARpT_s}}{1+\dfrac{0.622L}{c_pp}\dfrac{de_s}{dT}\left(1+\dfrac{ART_s}{0.622L}\right)} \qquad (3.4)$$

where e_s is the saturation vapor pressure, which is a function of T_s alone. For the water stage, e_s is given by

$$e_s=6.11\times\exp\left\{25.22\left(1-\frac{273}{T_s}\right)\right\}\left(\frac{273}{T_s}\right)^{5.31} \text{ in mb.} \qquad (3.5a)$$

Substituting this formula in (3.4) we obtain, approximately

$$\frac{dT_s}{d\zeta}=\frac{0.2876T_s}{\zeta}\,\frac{1+\dfrac{9.045Le_s}{pT_s}}{1+\dfrac{17950Le_s}{pT_s^2}\left(1-\dfrac{T_s}{1300}\right)}. \qquad (3.5b)$$

For a detailed derivation of this equation see Rossby (1932). The mixing ratio $q_s(T_s)$ is obtained from (2.9) and (3.5a) for $T=T_s$. Thus, when the temperature and the condensation levels are known, the vertical distributions of T_s and q_s inside the cloud can be determined from the Eqs. (3.5b), (3.5a) and (2.9).

Our next task is to find the rate of production of the cloud air. Towards this end we observe at first that the total rate of accession of moisture to a unit column of air through advection and through evaporation from below can be obtained by integrating Eq. (2.8) over the mass of the air-column and is given by

$$\frac{p_0}{g}I=\frac{p_0}{g}\left\{-\frac{1}{r}\frac{\partial}{\partial r}\int_0^1 rvqd\zeta+\frac{C_DV_0}{H^*}(q_e-q_0)\right\}, \quad (3.6)$$

where I stands for the quantity in the curly bracket, H^* equals RT_0/g, C_D is the drag coefficient, V_0 is the surface wind velocity, q_e is the mixing ratio of the underlying sea surface, and q_0 is the mixing ratio of the air at some reference level near the surface. Thus the first term on the right represents the accession of q by horizontal advection while the second term represents the accession of q by evaporation from below. (It may be mentioned that, in the region of the developing storm over water, the accession of moisture from below must be largely through wave-breaking, therefore q_0 must refer to a somewhat higher level.)

We now assume that all of this accession of water vapor, I, goes into making the cloud columns (T_s,q_s) from the environmental air (T,q). Part of I will condense (and precipitate), thereby raising the temperature of the air from $T(\zeta)$ to $T_s(\zeta)$, while the remainder will appear as increase in moisture content (i.e., latent energy) of the cloud column. The amount of moisture used in the first part of the process is determined by equating the latent heat of the vapor so condensed to the change in internal plus potential energy of the column:

$$\delta\bar{q}_{21}=\frac{c_p}{L}\int_{\zeta_t}^{\zeta_b}(T_s-T)d\zeta, \quad (3.7)$$

where ζ_t and ζ_b are the non-dimensional pressures at the top and bottom of the cloud. The amount of moisture used in the second part is simply given by

$$\delta\bar{q}_{22}=\int_{\zeta_t}^{\zeta_b}[q_s(T_s)-q]d\zeta. \quad (3.8)$$

Thus the total amount of water vapor needed for the creation of a model cloud over a unit area measured relative to the environmental air is given by the sum of these two integrals, viz.:

$$\begin{aligned}\frac{(p_b-p_t)}{g}\delta\bar{q}_2&=\frac{p_b-p_t}{g}(\delta\bar{q}_{21}+\delta\bar{q}_{22})\\ &\doteq\frac{p_0}{g}(\delta\bar{q}_{21}+\delta\bar{q}_{22}).\end{aligned} \quad (3.9)$$

Now, according to our assumption (i), deep cumulus convection always takes place in regions of conditionally unstable stratification and low-level convergence of moisture, and in such regions the release of latent heat is accomplished through deep cumulus convection. Therefore, the rate of production of the model cloud over a unit area in such a region is given by

$$\frac{l}{\Delta t}=\frac{I}{\delta\bar{q}_2}, \quad (3.10)$$

or the total production of such clouds in time Δt is given by

$$l=\frac{I\Delta t}{\delta\bar{q}_2}. \quad (3.10a)$$

In this latter form l may be interpreted somewhat figuratively by the following consideration. Before the atmosphere is completely saturated with cumulus cloud-air, we can always take Δt small enough so that $I\cdot\Delta t$ is less than $\delta\bar{q}_2$. In that case, the ratio may be interpreted as the fractional area of the sky that is covered by newly formed cumulus cloud as a result of the accession of moisture by advection and by evaporation from below. However, this interpretation of l demands l to be less than unity, which is actually not necessary for the validity of the relations we are going to establish. Thus we may just interpret l as the total amount of cloud air produced during the time Δt.

In view of the figurative interpretation of l as the percentage area coverage of the newly formed cumulus clouds, we may use l in place of the percentage area $b(z)$ in (3.3) and (3.4) and so obtain the changes of temperature and humidity resulting from the cumulus convection in time Δt, under the assumption that all the clouds are vertical and have uniform cross sections. It is then evident that we may also use $l/\Delta t$ in place of b in these equations and obtain the rate of change of temperature and humidity at any level ζ as the result of convection. In particular, the rate of accession of latent heat $Q_e(\zeta)$ to the mean air at the level ζ is given by

$$\begin{aligned}Q_e&=\frac{c_pl}{\Delta t}[T_s(\zeta)-T(\zeta)]\\ &=\frac{I}{\delta\bar{q}_2}c_p[T_s-T] \quad \text{for } I>0, \quad T<T_s \quad (3.11)\\ &=0 \quad \text{for } I\leqslant 0 \text{ or } T\geqslant T_s.\end{aligned}$$

Now we shall obtain the local change of the mean temperature field which results from the latent heat accession Q_e. According to our assumption (iv), the cumulus convection is accompanied by strong turbulence which is to bring about a complete mixing of the cloud-air with the surrounding air. Assuming that at time $t+\Delta t$ the temperature of the environmental air before mixing is T^*, then the mean temperature T after mixing is given by

$$\begin{aligned}T_{t+\Delta t}&=T^*+\Delta T_{Q_e}\\ &=T^*+l(T_s-T^*)\\ &=lT_s+(1-l)T^*,\end{aligned} \quad (3.12)$$

where ΔT_{Q_e} is the temperature increase resulting from the released latent heat. The second form of this equation indicates that ΔT_{Q_e} is the area average of the temperature change produced by latent heat accession.

Since T^* is the temperature of the mean air at $t+\Delta t$ when no latent heat is released, it is given by

$$T_{t+\Delta t}^* = T_t - \Delta t\left\{\frac{\partial(vTr)}{r\partial r} + \frac{\partial \omega T}{\partial \zeta} - \frac{\varkappa\omega T}{\zeta} - \nabla\cdot K\nabla T - \frac{Q_r}{c_p}\right\}. \quad (3.12a)$$

Therefore, the actual change of the mean temperature field is the sum of the adiabatic change plus the conductional change and the change due to the latent heat accession.

Similarly, we find that the value of q at time $t+\Delta t$ after mixing is given by

$$q_{t+\Delta t} = q^* + l[q_s(T_s) - q], \quad (3.13)$$

where q^* is the mixing ratio of the environmental air. It must be mentioned, however, that the q^*-computation differs from that of T^*, since the release of latent heat and the vertical transport of moisture are attributed to the deep cumulus convection alone. This means that q^* at any level of the cloud must be taken as that of the mean air at time t, and therefore the evaluation of the local rate of change of q through Eq. (2.8) is limited to the cloud free regions, i.e., below the cloud base and in the region of low level divergence.

Thus according to Eqs. (3.12), (3.12a) and (3.13), the changes of temperature and mixing ratio produced by deep cumulus convection at any point in the large scale system can be computed from the accession of moisture by advection and evaporation as given by (3.6) and the values of T_s and q_s of the moist adiabat.

We note that the quantity I of (3.7) is determined by the radial velocity v of the large scale circulation, especially that of the lowest layer, and that neither the mean ω nor the ω' of the convective motion appear in the determinations of Δq and ΔT_{Q_e}.

Another point worth mentioning is that only a part of the latent heat released by cumulus convection is available for raising the mean temperature of the air; the other part of the heat so released must be used to compensate for the cooling in expansion, which is represented by the terms involving ω in (3.12a) for the large scale flow. This is reflected in the fact that T and q of the large scale system increase with the accession of latent heat only when the atmosphere is not saturated with cumulus cloud air. When T and q have reached the values T_s and q_s of the moist adiabat, no further increase of temperature and humidity will result from additional accession of moisture, unless there is a change of the condensation level, even though latent heat of condensation is still being supplied. It can easily be shown that, after reaching this stage, the effect of the accession of latent heat Q_e in Eq. (3.12) is exactly cancelled by the terms of T^* involving ω, as given by Eq. (3.12a), since Q_e is now actually given by Eq. (2.10) with $q_s = q_s(T_s)$ and

$$\frac{\partial\theta}{\partial\zeta} = -\frac{L}{c_p\zeta^\kappa}\frac{\partial q_s}{\partial\zeta}.$$

That is to say, after this stage, the latent heat Q_e is used completely to compensate for the adiabatic cooling accompanying the large scale ascending motion and therefore no further local temperature change will result from this latent heat accession. Thus for a given condensation level, T_s and q_s of the moist adiabat represent the limiting values of T and q that can be approached by the release of latent heat through accession of moist air and cumulus convection. This is one of the distinct features of the present model which is not present in other models. This consideration shows clearly that intense hurricanes cannot be created nor maintained for long over land by inflow of normal unsaturated tropical atmosphere, since the condensation level is relatively high and T_s is not much higher than T. Only evaporation and heating from below can lower the condensation level and raise the limiting temperature T_s.

We mention again that our formulas (3.11), (3.12a) and (3.13) remain valid even for $l>1$, i.e., even after the atmosphere has become completely saturated with cumulus cloud air, since both the ratios $c_p(T_s-T)/(L\delta\bar{q}_2)$ and $(q_s(T_s)-q)/\delta\bar{q}_2$ and their vertical integrals are finite. In fact, the vertical integral of the former approaches 1 while that of the latter approaches zero as limits, indicating that all the net gain of humidity resulting from horizontal transport is precipitated out. It is evident that the ratio l can then no longer be interpreted as the fractional area of the cloud, but only as the rate of production of cloud air.

Thus our procedure for determining the changes of T and q may be summarized as follows: in the stable or descending and cloud free regions, we compute the changes of these two variables directly from Eqs. (2.6) and (2.8), whereas in the regions of conditionally unstable stratification and ascending motion we determine the change of q by (3.13) and add the temperature change ΔT_{Q_e} given by (3.12) to that obtained from (2.6) without Q_e. Thus in all these computations, unstable stratification does not appear directly in the equations. It is obvious that this method of including the effect of the small scale convections can be applied to all types of large scale circulations, and it is in no way limited to the hurricane problem investigated here.

One special character of the present approach is that latent heat is being released before the mean air is completely saturated, and is therefore very different from the case when latent heat is released by the large-scale ascending motion. As is to be expected, the stratification given by Eq. (3.5b) after saturation and mixing is more stable than the original super moist-adiabatic lapse-rate, since the convective motion transports heat upward. The ultimate limit is the moist-adiabatic.

Before leaving this subject, let us examine the vertical distributions of the quantities $\delta_p q$ and $\delta_c q$, the condensed and the latent water vapor in a deep cumulus tower:

$$\delta_c q = q_s(T_s) - q, \tag{3.8b}$$

$$\delta_p q = \frac{C_p}{L}(T_s - T). \tag{3.8c}$$

The moist adiabat used in these computations is the T_s-curve in Fig. 1, which is based on a value of 82% for the relative humidity of the surface air, or a mean relative humidity of 86% for the lowest 600 m layer ($\sim$60 mb). The vertical distributions of these two quantities so obtained are illustrated in Fig. 2. It is seen that $\delta_p q$ has its maximum at the 300-mb level, whereas $\delta_c q$ has its maximum at the 750-mb level, both for the normal condition and for the exceptional case with saturated surface air. From these vertical distributions of $\delta_p q$ and $\delta_c q$ we find that the value of the fraction $m = \delta\bar{q}_{21}/\delta\bar{q}_{22}$ for the normal tropical atmosphere is about 0.25, whereas for the exceptional case of saturated surface air it is about 0.33. These values indicate that in either case more than two-thirds of the moisture brought in by advection will be stored up in the atmosphere to increase its humidity and that less than a third is condensed and precipitated out in the tall cloud during the initial moment. The smallness of m obtained here is due to the low relative humidity of the normal tropical atmosphere, but even when the lower atmosphere is nearly saturated m will not be higher than 0.5, showing that it is always important to take into consideration the storage of moisture in computing the precipitation from moisture import. Only when the environmental air is completely saturated will m approach 1, as has been pointed out already.

It should be mentioned that the selection of the moist adiabat (and therefore T_s) by the property of the air of the lowest layer tends to overestimate $T_s - T$ at upper levels. A composite $(T_s - T)$ distribution can be obtained by using the net moisture flux in each elementary layer through the depth of the conditionally unstable layer of the atmosphere as a weighting factor. However, it does not seem worthwhile to introduce such a scheme at this stage as it would complicate the computations enormously.

4. Turbulent diffusion and flux through the boundaries

Assuming that the effects of the deep penetrating cumulus type convective motions created by thermal instability can be taken into consideration by the method described in Section 3, one may then treat the other transport terms represented by the various J's in (2.7) and J_q in (2.8) as eddy exchange terms and relate them to the gradients of the mean quantities so that:

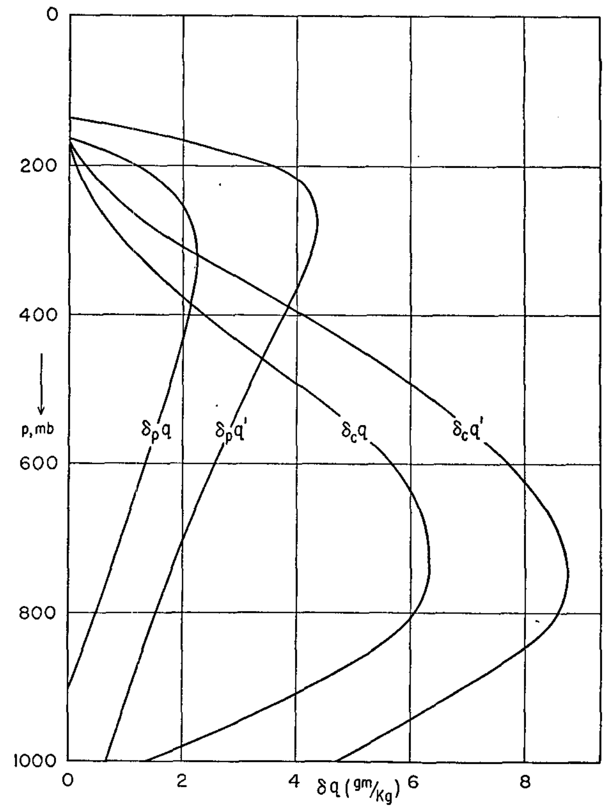

FIG. 2. Vertical variations of $\delta_p q$ and $\delta_c q$ for mean tropical air and for tropical air saturated at sea level.

$$\begin{aligned} -J_{u,v} &= \left\{ \frac{\partial}{\partial\zeta} K \frac{\partial}{\partial\zeta} + \frac{\partial}{\partial r} \frac{K_r}{r} \frac{\partial}{\partial r} r \right\} (u,v) \\ -J_T &= \frac{\partial}{\partial\zeta} A \frac{\partial T}{\partial\zeta} + \frac{1}{r}\frac{\partial}{\partial r}\left(r A_r \frac{\partial T}{\partial r} \right) \\ -J_q &= \frac{\partial}{\partial\zeta} B \frac{\partial q}{\partial\zeta}. \end{aligned} \tag{4.1}$$

Horizontal diffusion of water vapor has been omitted. Here, the various vertical exchange coefficients are related to the corresponding coefficients in z-coordinate by the formulae

$$(K,A,B) = \left(\frac{g}{RT_0} \frac{\rho}{\rho_0} \right)^2 (K_z, A_z, B_z). \tag{4.2}$$

For simplicity and also because of lack of strong evidence to the contrary, we shall take the three coefficients K_z, A_z, B_z for momentum, heat and water vapor as to be equal and of the order of 10^5 to 10^6 cm^2 sec^{-1}, in accordance with the best estimates available at present (see Riehl and Malkus, 1961). Thus we shall take

$$K=A=B=(10^{-7}-10^{-6})(\rho/\rho_0)^2 \text{ sec}^{-1}. \quad (4.2\text{a})$$

The precise value may be varied within these limits.

It may be mentioned that in Eq. (4.1) we have assumed that the vertical turbulent heat transfer is proportional to the mean vertical gradient of temperature instead of that of the potential temperature, in order to get an upward heat flux across the lower surface. This formulation is consistent with the assumption that the turbulence under consideration is of thermal origin.

Of the horizontal exchange coefficients we know very little except that they are often neglected in practical analysis. However, we shall find that they are important in many ways, especially in the inner part of the system where large horizontal gradients develop. Further, since the statistical influence of the deep cumulus convection is taken as an integral part of the model, the eddy transport coefficient should be somewhat higher than when such motions are absent. For simplicity, we choose the constant value

$$K_r=A_r=10^4 \text{ m}^2 \text{ sec}^{-1}. \quad (4.3)$$

This corresponds in Richardson's empirical formula to a horizontal eddy scale of 33 km, which does not seem unduly large. On the other hand, it is also reasonable to experiment with exchange coefficients proportional to the mean velocity, since all these eddy exchange terms are nonlinear.

At the lower boundary, which we shall take as $\zeta=1$, we adopt the customary form of the surface wind stresses, given by

$$(\tau_{\lambda,0},\tau_{r,0})=-\frac{p_0}{g}\left[K\frac{\partial}{\partial\zeta}(u,v)\right]_{\zeta=1} \quad (4.4)$$
$$=C_D\rho_0 V_0(u_0,v_0),$$

where u_0, v_0 are the velocity components at the surface, and C_d is the drag coefficient. The value of C_d will be taken as 2.5×10^{-3} on the average, even though some observations show that C_d itself has a tendency to increase linearly with the velocity of the mean wind (Miller, 1963).

On the other hand, we shall depart from the customary assumption that the eddy stresses vanish at high levels. Judging from the fact that small scale convective motions are very active at those levels, and that the vertical profile of the radial velocity is nearly symmetric with respect to the mid-level of the active troposphere, it is also reasonable to assume the existence of a non-vanishing eddy stress at the level $\zeta=\zeta_1$, given by

$$\frac{RT_0 p}{g}K\frac{\partial}{\partial\zeta}(u,v)=\gamma\rho V_1(u_1,v_1) \quad (4.5)$$

so that there may also be a drag on the layer below ζ_1, if the wind velocities increase downward. The effect of such high level diffusions can be investigated by varying γ.

For the heat flux and evaporation from the ocean surface, we assume

$$\begin{aligned}\frac{p_0}{g}\left(A\frac{\partial T}{\partial\zeta}\right)_{\zeta=1}&=\rho_0 C_d V_0(T_e-T_0)\\ \frac{p_0}{g}\left(B\frac{\partial q}{\partial\zeta}\right)_{\zeta=1}&=\rho_0 C_d V_0(q_e-q_0),\end{aligned} \quad (4.6)$$

where T_e and q_e are the temperature of the ocean surface and the corresponding saturation mixing ratio, and T_0 and q_0 are the temperature and mixing ratio of the air at a reference level.

When these conditions are taken into consideration, we find that the kinetic and thermal energy integrals can be written as follows:

$$\frac{\partial K}{\partial t}+D_i+D_s=-R\int_m\frac{\omega T}{\zeta}dm, \quad (4.7)$$

$$\frac{\partial E}{\partial t}=\int_m\left(Q+\frac{R\omega T}{\zeta}\right)dm+H_s, \quad (4.8)$$

where K is the total kinetic energy of the horizontal motion, E is the total enthalpy, D_i is the internal dissipation of the kinetic energy, D_s is the dissipation by skin friction,

$$D_s=\int\int\left[\rho_0 C_d V_0{}^2+\frac{p_0}{2g}\frac{\partial V^2}{\partial\zeta}\bigg|_{\zeta=1}\right]dxdy, \quad (4.8\text{a})$$

and H_s stands for the total flux of sensible heat through the boundaries.

Eq. (4.7) shows that the increase of the total horizontal kinetic energy must come from a negative correlation between ω and T of the mean field, whereas Eq. (4.8) shows that this correlation is maintained by the release of latent heat and the flux of sensible heat through the boundaries.

5. A balanced dynamic model and its two-level approximation

Even though the system of Eqs. (2.2)-(2.8) has already been simplified enormously by the elimination of asymmetry and departures from hydrostatic balance, they still permit internal gravity waves, and their numerical integration must therefore be carried out in relatively short time steps. In order to remove these internal gravity waves from the present problem, we assume that the radial pressure gradient is in permanent balance with the Coriolis force and the centrifugal force associated with the tangential motion, except in the Ekman boundary layer. Thus Eq. (2.3) simplifies to

$$0=-\frac{\partial\phi}{\partial r}+\left(f+\frac{u}{r}\right)u. \tag{5.1}$$

Because of the balance requirements (5.1) and (2.4), we can no longer make time-extrapolations for v and ω through the radial and the vertical equations of motion. Instead, these two quantities must now be obtained through a diagnostic relation that is consistent with (5.1) and (2.4). To derive this relation, we eliminate ϕ from these two equations and obtain the following thermal wind relation

$$\left(f+\frac{2u}{r}\right)\frac{\partial u}{\partial\zeta}=-R\zeta^{x-1}\frac{\partial\theta}{\partial r}. \tag{5.2}$$

By taking the time derivative of this equation, substituting $\partial^2u/\partial t\partial\zeta$, $\partial u/\partial t$ and $\partial^2\theta/\partial r\partial t$ from the ζ-and the r-derivatives of (2.2) and (2.6) and introducing the stream-function ψ defined by

$$v=-\frac{\partial\psi}{\partial\zeta},\quad \omega=\frac{\partial\psi}{\partial r}+\frac{\psi}{r} \tag{5.3}$$

we obtain the following linear non-homogeneous equation for ψ:

$$R\zeta^{x-1}\frac{\partial\theta}{\partial\zeta}\frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial r}+\frac{\psi}{r}\right)-C\frac{\partial^2\psi}{\partial\zeta^2}-R\zeta^{x-1}\frac{\partial\theta}{\partial r}\left[2\frac{\partial^2\psi}{\partial r\partial\zeta}-\frac{2}{r}\frac{\partial\psi}{\partial\zeta}-\frac{1-x}{r\zeta}\frac{\partial\psi r}{\partial r}\right]=H. \tag{5.4}$$

Here

$$C=\left(f+\frac{2u}{r}\right)\left(f+\frac{\partial ur}{r\partial r}\right)$$

and H is given by

$$\begin{aligned}H&=\frac{R}{c_p\zeta}\frac{\partial Q}{\partial r}+\frac{\partial}{\partial\zeta}\left[\left(f+\frac{2u}{r}\right)J_u\right]-\zeta^{-x}\frac{\partial J_T}{\partial r}\\&=\frac{R}{c_p\zeta}\frac{\partial Q}{\partial r}+\frac{\partial}{\partial\zeta}\left\{\left(f+\frac{2u}{r}\right)\left[\frac{\partial}{\partial\zeta}(K-A)\frac{\partial u}{\partial\zeta}+(K_r-A_r)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)\right]\right.\\&\qquad\left.-\frac{2}{r}\left[A\left(\frac{\partial u}{\partial\zeta}\right)^2+A_r\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right)^2\right]\right\}-A\left(f+\frac{2u}{r}\right)\frac{\partial u}{\partial\zeta}.\end{aligned} \tag{5.4a}$$

Thus, when the spatial distributions of u,θ and the non-adiabatic heating Q are known at any moment, we can solve (5.4) for ψ with the proper boundary conditions.

We mention that in arriving at these simple versions the relation (5.2) has been used consistently. It is seen that when $A=K$ and $A_r=K_r$, the u terms in H largely cancel each other except in the Ekman boundary layer, where the surface stress $\tau_{\lambda,0}$ will appear in the K term. Thus near the boundary H is determined largely by the stress $\tau_{\lambda,0}$ whereas in the free atmosphere it is determined mainly by the heating functions Q_e and Q_r. Eq. (5.4) is similar to the equations used by Kuo (1954, 1956a, b) and Eliassen (1952) in their discussions of axially symmetric motions under frictional and thermal influences. A similar equation has also been employed by Estoque (1963) in computing the meridional circulation in a steady state mean hurricane. However, he used Eq. (2.10) for evaluating Q_e whereas we do not.

For the present investigation, we shall simply choose the two isobaric surfaces $p=p_0=1000$ mb and $p=p_1=0$ or 100 mb as two boundaries on which $\psi=$constant, and the central line $r=0$ on which $v=0$. Thus on all these three boundaries we shall have $\psi=$constant$=0$. (A more realistic lower boundary condition would be to require ψ to vanish on the surface $p_0=p_0(r)$, which may be determined from the temperature distribution.) As a fourth boundary we take the surface defined by a large value of r, (r_2, say) and there assume either the vanishing of the horizontal divergence and therefore of ω at all levels (which corresponds to an open system) or the vanishing of v (which corresponds to a closed system). Thus the boundary conditions we impose on ψ are

$$\psi=0\quad\text{at }\zeta=1,\ \zeta=\zeta_1=0\text{ or }0.1,\text{ and }r=0,\text{ and} \tag{5.5}$$

$$\frac{\partial\psi}{\partial r}+\frac{\psi}{r}=0\quad\text{at }r=r_2\text{ for an open system or} \tag{5.5a}$$

$$\psi=0\quad\text{at }r=r_2\text{ for a closed system.} \tag{5.5b}$$

The solution of (5.4) and the boundary conditions (5.5) and (5.5a) or (5.5b) is unique if, and only if

(5.4) is of elliptic type, i.e., if

$$R\zeta^{x-1}\left(\frac{\partial\theta}{\partial r}\right)^2+C\frac{\partial\theta}{\partial\zeta}<0. \qquad (5.6)$$

This is also the condition for non-existence of free meridional circulations (see Kuo, 1954, 1956). Since in our model the release of latent heat is attributed to small scale convection, the mean stratification will be permanently stable; therefore the condition (5.6) requires that C be positive and $\partial\theta/\partial r$ be less than a limiting value, corresponding to dynamic and baroclinic stability. Only under such conditions can the balanced model be expected to hold. One of the interesting points of the numerical integration of this model is to find out whether (5.6) will continue to be satisfied during the course of development.

It may be mentioned that if the balance condition (5.2) is assumed to be satisfied exactly, then, time extrapolations of u and θ cannot be obtained independently, except at the boundary. That is to say, we must either obtain u by (2.2), or obtain θ by (2.6), and determine the other through integrating (5.2). Since in the area of saturation the vertical distribution of θ must be determined by Eq. (3.12) and since the supply of sensible and latent heat is the main energy source for the development, it seems more convenient to obtain θ directly from (3.12) and to obtain u by integrating (5.2) with respect to ζ.

On the other hand, we may also interpret (5.2) as an approximate relation sufficiently accurate for Eq. (5.4) to hold, but not so accurate as to exclude the independent calculations of the time rates of changes of u and θ from the respective prognostic equations. Such a procedure is justifiable by a first order expansion of the variables in terms of a small parameter ϵ, if u and θ are of zero order and v and w are of first order in ϵ.

Thus beginning with a given initial state, time integration of the model consists of a repetition of the following three steps: (i) from the known distributions of u and θ we at first compute the forcing function H from (5.4a) and the boundary flux of momentum and energy; (ii) we then solve for ψ from (5.4) and the boundary conditions (5.5) and (5.5a) or (5.5b), and (iii) compute the future values of u in the boundary layer from Eq. (2.2), and the future values of θ and q at all points from (3.12), (2.8) and (3.13), using the ψ obtained in step (ii). These three steps complete a cycle of computations for the time-extrapolation of the dependent variables ψ, u, θ and q.

It may be noted that in solving for ψ with the boundary conditions (5.5) we have used Eq. (5.4) not only for the free atmosphere, but also for the boundary layer, which may seem objectionable. Alternatively, we may set aside an Ekman boundary layer and solve (5.4) with the lower boundary condition $\psi_e=(\zeta-\zeta_e)v_b$ at the top of the friction-layer ζ_e, where v_b is the mean boundary layer radial velocity. v_b may be obtained either directly from (2.3) or by a simple approximation of Eq. (2.2).

In order to investigate the characteristics of this model without too complicated computations, we shall at first make use of a two-level representation of the vertical distributions of the various quantities. Let us denote the quantity χ at the levels $\zeta=n(\zeta_4-\zeta_0)/4(n=0,1,2,3,4)$ by χ_n, where $\zeta_4(=\zeta_e)$ and ζ_0 are the values of the nondimensional pressure at the top of the friction-layer and the top of the disturbance, respectively. We shall take $\zeta_0=1-\zeta_4=0.1$ or zero, depending upon whether or not a separate Ekman layer is assumed, and either $\psi_0=0$ or $\psi_0=\psi_4$.

In this two-level approximation, u, v, ω, θ and q can be specified at the levels ζ_1 and ζ_3 provided certain rules of interpolation are adopted. Alternatively, we may also deal with the sums and differences of the various quantities at these two levels, viz.:

$$\begin{aligned}
&\bar{u}=\tfrac{1}{2}(u_1+u_3), \qquad \Delta u=u_3-u_1\\
&\bar{\theta}=\tfrac{1}{2}(\theta_1+\theta_3), \qquad \sigma=\tfrac{1}{2}(\theta_1-\theta_3)\\
&v_3=\frac{1}{\Delta\zeta}(\psi_2-\psi_4), \qquad v_1=-\frac{1}{\Delta\zeta}(\psi_2-\psi_0), \qquad \Delta\zeta=\zeta_{k+1}-\zeta_{k-1}.
\end{aligned} \qquad (5.6)$$

Within the accuracy of the two-level representation, $\bar{u}$ and $\bar{\theta}$ may be interpreted as the u and θ at the middle level ζ_2, even though somewhat different expressions may result from the different interpretations. Thus we have

$$\begin{aligned}
\frac{\partial\bar{u}}{\partial t}&=-\frac{1}{2r^2}\frac{\partial}{\partial r}[r^2(u_1v_1+u_3v_3)]-\frac{f}{2}(v_1+v_3)-\frac{1}{2\Delta\zeta}(\omega_4u_4-\omega_0u_0)-\bar{F}\\
&=-\frac{1}{4\Delta\zeta r^2}\frac{\partial}{\partial r}[r^2(2\psi_2-\psi_4-\psi_0)\Delta u]+\frac{(\psi_4-\psi_0)}{2\Delta\zeta}\left[f+\frac{\partial\bar{u}r}{r\partial r}\right]-\frac{1}{2\Delta\zeta}\{\omega_4(u_4-\bar{u})+\omega_0(\bar{u}-u_0)\}-\bar{F},
\end{aligned} \qquad (5.7)$$

where $\bar{F}$ is the mean tangential frictional force. In the absence of friction ($\bar{F}=\psi_0=\psi_4=0$), this equation satisfies the conservation of absolute angular momentum of the system.

On making use of the approximations $\partial\theta/\partial\zeta)_1=\partial\theta/\partial\zeta)_3=-2\sigma/\Delta\zeta$, the equation for $\partial\bar{\theta}/\partial t$ may be written as follows:

$$\frac{\partial\bar{\theta}}{\partial t}=\frac{1}{2\Delta\zeta r}\frac{\partial}{\partial r}[r\sigma(2\psi_2-\psi_4-\psi_0)+r\bar{\theta}(\psi_4-\psi_0)]+\frac{\sigma}{\Delta\zeta}(\omega_4+\omega_0)-\frac{1}{2\Delta\zeta}\bar{\theta}(\omega_4-\omega_0)+N_1+G_1. \tag{5.8}$$

When $N_1=G_1=0$ and either $\omega_4=\omega_0=0$ or $\sigma=$constant, this equation satisfies the conservation of the mean $\bar{\theta}$. Alternatively, we may also obtain $\partial\theta_2/\partial t$ directly from (2.6), which for the present case takes the form

$$\frac{\partial\theta_2}{\partial t}=\frac{(\psi_4-\psi_0)}{\Delta\zeta}\frac{\partial\theta_2}{\partial r}+\frac{2\sigma\omega}{\Delta\zeta}+N_1+G_1. \tag{5.8a}$$

This equation satisfies the conservation of the mean θ_2 when σ is a constant and $\psi_4-\psi_0$, N_1 and G_1 are zero. Similarly, by taking the difference between $\partial\theta_1/\partial t$ and $\partial\theta_3/\partial t$ we obtain

$$\frac{\partial\sigma}{\partial t}=\frac{1}{2\Delta\zeta}\left\{(2\psi_2-\psi_4-\psi_0)\frac{\partial\bar{\theta}}{\partial r}+(\psi_4-\psi_0)\frac{\partial\sigma}{\partial r}\right\}+N_2+G_2. \tag{5.9}$$

Here the quantities N_1, N_2, G_1, G_2 are defined by

$$N_1=\frac{1}{2c_p}\left\{\frac{Q_{e1}}{\zeta_1^{\kappa}}+\frac{Q_{e3}}{\zeta_3^{\kappa}}\right\},\qquad N_2=\frac{1}{2c_p}\left\{\frac{Q_{e1}}{\zeta_1^{\kappa}}-\frac{Q_{e3}}{\zeta_3^{\kappa}}\right\}.$$

$$G_1=C_0\delta T_0+\frac{1}{r}\frac{\partial}{\partial r}\left(Br\frac{\partial\bar{\theta}}{\partial r}\right),\quad G_2=-C_0\delta T_0+\frac{1}{r}\frac{\partial}{\partial r}\left(Br\frac{\partial\sigma}{\partial r}\right), \tag{5.10}$$

where

$$C_0=\frac{gC_D}{RT_0}V_0,\quad \delta T_0=T_e-T_0.$$

From the discussions in Section 3 we know that T_s represents the final equilibrium temperature. However, in the calculation this temperature can only be approached through very small time steps. In order to avoid such calculations we shall impose the conditions

$$\frac{\partial\theta_2}{\partial t}=0 \quad\text{and}\quad \frac{\partial\sigma}{\partial t}=0$$

for every point where θ_2 has approached or exceeded θ_{s2}.

The heating function Q_e is approximated semilinearly in this two-level representation, in accordance with the normal distribution exhibited in Fig. 2, by setting

$$Q_{e2}=2Q_{e3},\quad Q_{e1}=3Q_{e3},\quad Q_{e,0.1}=Q_{e,.9}=0 \tag{5.10a}$$
$$Q_e=Q_e \text{ max. at } \zeta=0.3$$

in the expressions N_1 and N_2 in (5.10). This yields

$$N_1=\frac{l}{\Delta t}\frac{(T_{s2}-T_2)}{4}\left\{\frac{1}{\zeta_1^{\kappa}}+\frac{1}{\zeta_3^{\kappa}}\right\}\doteq 1.135\frac{l}{\Delta t}(\theta_s-\theta) \tag{5.10b}$$

$$N_2\doteq 0.30\frac{l}{\Delta l}(\theta_s-\theta),$$

where θ and θ_s are the environmental and cumulus cloud potential temperature at level $\zeta_2=0.5$. The two constants are based on $\zeta_1=0.25$ and $\zeta_3=0.75$.

To calculate the factor $l/\Delta t$ as defined by (3.10) in this two-level representation, we make use of the linear approximations

$$v=2v_0(\zeta-0.5) \tag{5.11a}$$

$$\begin{aligned}T_s-T&=2(1-\zeta)(T_{s2}-T_2) \quad \text{for} \quad 0.25<\zeta<1\\ &=15(\zeta-0.15)(T_{s2}-T_2)\\ &\qquad \text{for} \quad 0.15<\zeta<0.25.\end{aligned} \tag{5.11b}$$

$$\begin{aligned}q&=1.25q_0(\zeta-0.2) \quad \text{for} \quad 0.2<\zeta<1\\ &=0 \quad \text{for} \quad \zeta<0.2\end{aligned} \tag{5.11c}$$

and an identical relation for $q_s(T_s)$, where v_0 and q_0 are the surface values of v and q.

On substituting (5.11a) and (5.11c) in the integral I of (3.7) we obtain

$$I=-\frac{2}{9r}\frac{\partial}{\partial r}(v_bq_br)+C_0\delta q_0, \tag{5.12}$$

where v_b and q_b are for the level $\zeta=0.95$, and $\delta q_0=q_e-q_0$.

For the quantity $\delta\bar{q}_2$ of (3.10) we make use of the linear approximation

$$q_s(T_s)-q=q_s(T_s)-hq_s(T)$$
$$\doteq q_s(T_s)\left\{1-h\exp\left[-\frac{273}{TT_s}(T_s-T)\right]\right\} \quad (5.13)$$
$$\doteq q_s(T_s)[1-h+h\alpha(T_s-T)],$$

where h is the mean relative humidity and $\alpha=273/TT_s=0.0042$ (deg C)$^{-1}$. Substituting (5.11b) and (5.13) in (3.9) we find

$$\delta\bar{q}_2=2.55\times10^{-4}(T_{s2}-T_2)[1+3.28hq_s(T_{s0})]+0.4(1-h)q_s(T_{s0}) \quad (5.14)$$
$$=2.09\times10^{-4}(\theta_s-\theta)_2[1+3.6hq_b]+0.436(1-h)q_b,$$

where T_{s0} is the moist adiabat temperature at surface level. Thus the quantities in Q_e can be expressed in terms of $(\theta_s-\theta)$ at the level ζ_2 and q_b and h.

The temperature $T_{s2}(=0.816\theta_{s2})$ of the moist adiabat at level ζ_2, which occurs in N_1 and N_2 both explicitly and implicitly through l, can be obtained from T_b and the condensation temperature T_{cb} of the mean surface air through the following approximate equation:

$$T_{s2}=0.41T_b+0.59T_{cb}-24.5\text{K}. \quad (5.15)$$

The derivation of this relation and the formula for calculating T_{cb} from T_b and q_b are given in the Appendix.

We mention that, even with a high relative humidity $h=0.9$ and $q_b=0.018$, the second term of $\delta\bar{q}_2$ is still comparable to a temperature difference of 4C and is therefore not negligible.

It is interesting to note that, if we disregard the last term of δq_2 and neglect $C_0\delta q_0$ from I and treat q_b as a constant, for example, $q_b=0.018$, we then obtain from (5.12), (5.14) (3.10) and (5.10c),

$$N_1=-\frac{1.135\times2.22q_b\omega_4}{2.2\times10^{-4}}=-0.2\tilde{\omega}_4, \quad \text{if } \omega_4<0$$
$$=0 \quad \text{if } \omega_4>0, \quad (5.10c')$$

where $\tilde{\omega}_4=dp/dt=p_0\omega_4$. This corresponds to Ogura's heating function (1964) with $\eta=3.3$ if we take $-\partial\bar{\theta}/\partial p=60$ C per 1000 mb, which is highly unstable. On the other hand, when the last term of $\delta\bar{q}_2$ is included and with $h\leqslant0.8$, the effective η-value is about 1, which is stable. Thus this consideration shows clearly that the behavior of the system depends very critically on the moisture content of the air.

It is easy to show that the use of the approximation (5.10c′) in conjunction with the boundary layer inflow represented by (5.19) below will lead to unlimited growth. Further, as ω_4 is apt to be proportional to ω_2, the characteristics of the model based on (5.10c′) are very likely to be similar to that of the model based on the conventional relation (2.10) for Q_e.

Since low-level convergence of moisture is needed in the evaluation of the quantity I of (5.12), and the low level temperature is needed in the determination of the condensation level temperature, q_b and θ_b for the level $\zeta_b=0.95$ were also calculated from the prognostic equations, together with u_b, which are considered as the mean values for the lowest 100-mb layer. The equations for these quantities are

$$\frac{\partial\theta_b}{\partial t}=-v_b\frac{\partial\theta_b}{\partial r}-4\omega_b(\theta_b-\theta+\sigma)+10C_0\delta T_0-\left(\frac{A}{\zeta^\kappa}\frac{\partial T}{\partial\zeta}\right)_{\zeta=0.9}+\frac{1}{r}\frac{\partial}{\partial r}\left(A_sr\frac{\partial\theta_b}{\partial r}\right), \quad (5.15)$$

$$\frac{\partial q_b}{\partial t}=-v_b\frac{\partial q_b}{\partial r}-1.3334\omega_bq_b+10C_0\delta q_0, \quad (5.16)$$

$$\frac{\partial u_b}{\partial r}=-v_b\left[f+\frac{\partial u_br}{r\partial r}\right]-\frac{2}{0.9}\omega_b(u_b-u_2)-\alpha C_0u_b+\frac{\partial}{\partial r}\left(\frac{K_s}{r}\frac{\partial u_br}{\partial r}\right), \quad (5.17)$$

where the linear extrapolations

$$\left.\frac{\partial\theta}{\partial\zeta}\right)_{\zeta_b}=20(\theta_b-\theta_4)=5(\theta_4-\theta_3)=4(\theta_b-\theta+\sigma)$$

$$\left.\frac{\partial u}{\partial\zeta}\right)_{\zeta_b}=20(u_b-u_4)=\frac{20}{9}(u_b-u_2)$$

have been used. For simplicity, the eddy transfer term $[K(\partial u/\partial\zeta)]_{\zeta=0.9}$ in (5.17) have been omitted and the factor 10 of the surface friction term replaced by α, which we assume to range from 1 to 10.

The two-level representation of the balance equation (5.4) is obtained by setting

$$\left.\frac{\partial\psi}{\partial\zeta}\right)_{\zeta=\frac{1}{2}}=\frac{1}{\Delta\zeta}(\psi_3-\psi_1)=\frac{1}{2\Delta\zeta}(\psi_4-\psi_0),\quad \left.\frac{d^2\psi}{d\zeta^2}\right)_{\zeta=\frac{1}{2}}=\frac{1}{(\Delta\zeta)^2}[\psi_4+\psi_0-2\psi_2],$$

where ψ_4 is the stream function at the top of the friction layer:

$$\frac{d^2\psi}{d\eta^2}+\left(\frac{1}{\eta}-\frac{0.714\Delta\zeta}{\sigma}\frac{d\theta}{d\eta}\right)\frac{d\psi}{d\eta}-\left[\frac{1}{\eta^2}-\frac{2120C}{\sigma\Delta\zeta}+\frac{0.714\Delta\zeta}{\eta\sigma}\frac{d\theta}{d\eta}\right]\psi+\frac{1063}{\sigma}\left\{H\Delta\zeta+\frac{C(\psi_4+\psi_0)}{\Delta\zeta}-\frac{470(\psi_4-\psi_0)}{\eta}\frac{d\theta}{d\eta}\right\}=0, \quad (5.18)$$

where $\eta=r\ \mathrm{km}^{-1}$, and ψ, θ, σ and C refer to their values at the level ζ_2.

In this formulation of the problem, it is customary to obtain the frictional stream function ψ_4 by assuming balance between the horizontal advection term and the surface friction term in Eq. (5.17), thereby giving

$$\psi_4=0.1v_b=-\frac{C_0u_b}{(Z_b)}, \quad (5.19)$$

where $Z_b=(f+\partial u_b/\partial r+u_b/r)$ is the absolute vorticity of the tangential flow. Since balance is assumed between the advection term and friction in the tangential equation of motion in this approach, the calculation of $\partial u_b/\partial t$ must be replaced by the calculation of $\partial u/\partial t$ for another level, preferably for the level ζ_3, and use u_3 for u_b in (5.19).

A very serious difficulty of this approach is the over-sensitive dependence of v_b and ψ_4 on Z_b, especially just outside of the radius of maximum u_b where $\partial u_b/\partial r$ is negative and when u_b is large, such that Z_b is either very small or even becoming negative. Eq. (5.19) then gives rise to very large and rapidly changing v_b which will lead to unmanageable developments. There does not seem to be any satisfactory way of removing this difficulty at the moment, which also reflects our lack of real understanding of the so called frictional flow theory. In the present investigation this difficulty will be controlled, somewhat artificially, by assigning a coefficient b to the $\partial u_b/\partial r$ term in Z_b, and taking either $b=1$ or $b=0$, even though it is hard to interpret the latter case rationally, except that it insures a surface layer inflow in the whole range of the cyclonic vortex.

Alternatively, we may omit ψ_4 completely, assuming (5.4) and thereby (5.18) is valid for the whole atmosphere. The surface layer Eqs. (5.15)-(5.17) are integrated by making use of the ψ_b obtained from extrapolating ψ linearly from ζ_2 to ζ_b. Since no frictional flow is present in this approximation, u_b and v_b should be compared with u_3 and v_3 of the approximation using (5.19).

As it is almost impossible to formulate a sound compromise between these two extremes within the framework of this two-level representation, these two schemes will be investigated separately through numerical integrations, with a view to the clarification of the respective possibilities and difficulties of these two schemes and to the finding of ways of improvement toward the formulation of a better approach.

6. Results of numerical integrations of the two-level balanced model

The Eqs. (5.7)-(5.9) and (5.15)-(5.18) have been integrated numerically. For economy a non-uniform grid in η was used, defined by

$$\eta=j(\tfrac{1}{2}j+3.5),\quad j=0,1,2,\cdots,44, \quad (6.1)$$

This covers the area from $r=0$ to 1170 km. For the solution of the ordinary differential equation (5.18) the conventional method is used first and later on the method described in Richtmyer's book (1957, p. 103) is employed. It is found that for the present problem the conventional method is very suitable.

Since our main purpose in these computations is to test the proposed method of including the latent heat input through deep cumulus convection and the various aspects of the balanced model, the following two sets of somewhat different formulations of the model have been investigated:

I. A two-level balanced model without a separate frictional layer, which is characterized by $\psi_4=0$ and $\Delta\zeta=0.5$ in (5.18).

II. A two-level balanced model with a separate friction layer, characterized by (5.19) for ψ_4 and $\Delta\zeta=0.4$.

Six different cases have been investigated for group I, namely:

Ia. Mid-level temperature from (5.8a) and thermal wind for Δu and u_b-u_2 in (5.7) and (5.17), $\alpha=1$ in (5.17);

Ib. Mid-level temperature from (5.8a) and Δu and u_b-u_2 directly from the u_b and $\bar{u}$ calculations, $\alpha=1$.

Ic. Mid-level temperature from (5.8a), thermal wind for Δu and u_b-u_2, $\alpha=1$, and $\delta T_0=0$, $A=0$. (No surface heat supply and no vertical diffusion of heat.)

Id. Mean-layer temperature from (5.8) and Δu and u_b-u_2 from thermal wind relation; $\alpha=1$, $\delta T_0\neq 0$.

Ie. Mid-level temperature from (5.8a), thermal wind for Δu and u_b-u_2, and $\alpha=10$.

If. Same as Ie except $\alpha=5$.

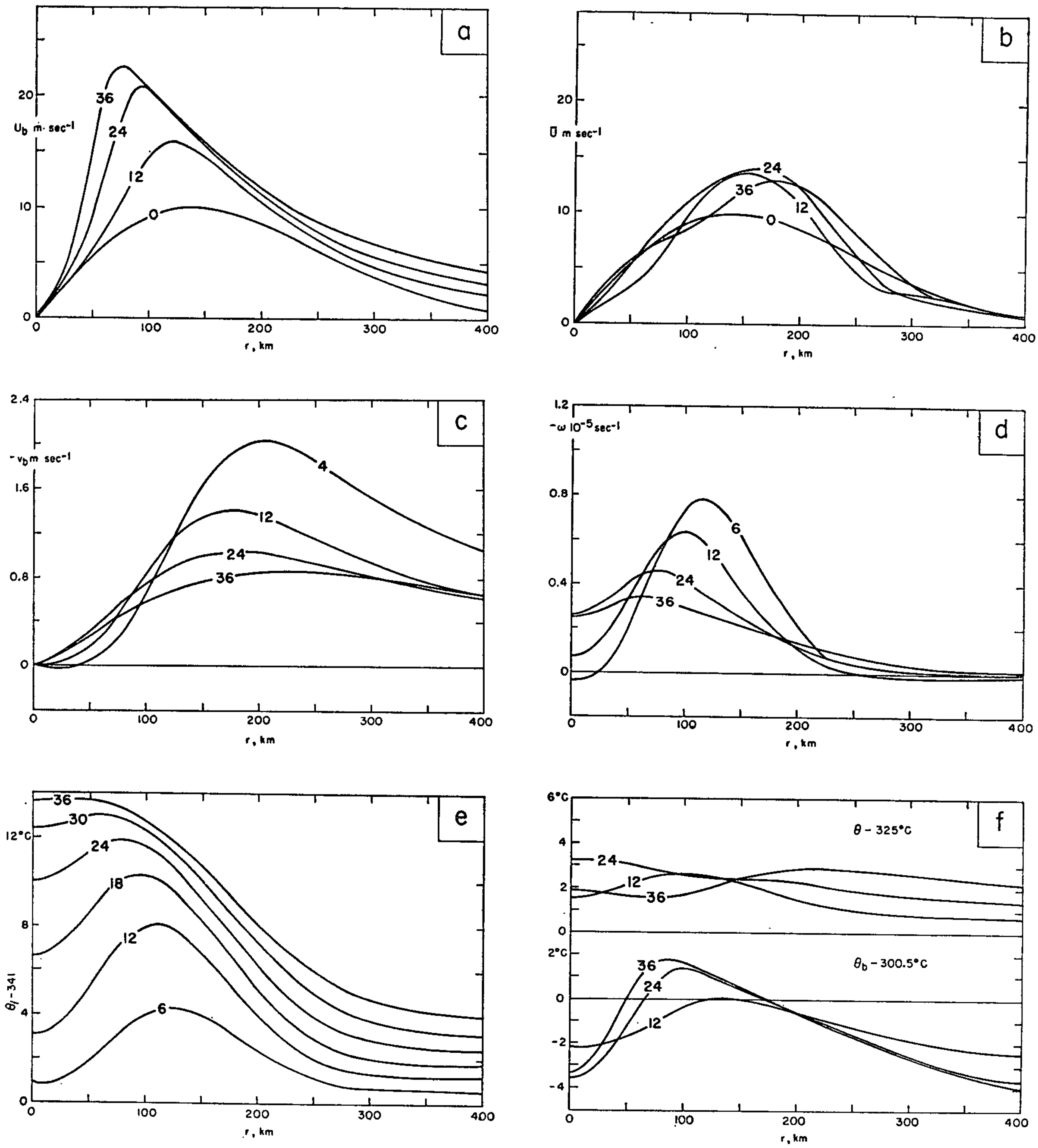

FIG. 3. Predicted time and space variations in experiment Ia of various quantities; attached numbers on curves indicate time in hours.

a) Low level tangential velocity u_b
b) Mid-level tangential velocity $\bar{u}$
c) Low level radial velocity v_b
d) Vertical ζ-velocity ω
e) Upper level potential temperature θ_1
f) Middle-level and low-level potential temperatures θ and θ_b.

A number of different cases have been investigated in the group II experiments, characterized either by $b=0$ or $b=1$ for the coefficients of du_b/dr in Z_b of (5.19) and by different heating relations. These different cases suffice to demonstrate the effects of the various approximations, especially with regard to (i) the mid-level temperature ($\bar{\theta}$) calculation as against the mean-layer temperature (θ_2) calculation, (ii) the use of thermal wind as against Δu, (iii) the effect of surface friction on u_b and (iv) the effect of a superimposed frictional circulation represented by ψ_4.

For the time integrations of Eqs. (5.7)-(5.9) and (5.15)-(5.17), centered time and space differences are used for group I and a forward time difference and an upstream space difference are used for group II. The time interval used in all these computations is two (2) minutes, which satisfies the stability requirement for all cases.

The initial state for all these integrations is represented by a weak barotropic circular vortex, characterized by

$$u=14.142\left(\frac{r}{L}\right)\exp\left\{\frac{1}{2}-\left(\frac{r}{L}\right)^2\right\},\ \text{in m sec}^{-1}, \tag{6.2}$$

with $L=200$ km, and $\partial u/\partial\zeta=0$ everywhere. The maximum tangential velocity of this vortex is 10 m sec^{-1} and is located at a distance of 141.42 km from the center.

The initial distributions of potential temperature and humidity are

$$\theta_b=300.5\text{C},\quad \theta_2=325.0\text{C},\quad q_b=0.0171, \tag{6.2a}$$
$$\sigma=16.0\text{C for I}\quad\text{and}\quad \sigma=12.8\text{C for II}.$$

The fluxes of heat and moisture through the lower boundary are characterized by

$$\begin{aligned}&\delta T_0=(301.0-0.98540\theta_b)\ \text{in deg C}\\ &\delta q_0=0.024-1.0667q_b\\ &C_0=V_0(1+.084V_0)\times10^{-7}\ \text{sec}^{-1},\ V_b\ \text{in m sec}^{-1}.\end{aligned} \tag{6.3}$$

The value of the eddy diffusion coefficient K at the surface used is 2×10^{-7} sec^{-1} or zero. For u_b, the effective friction is taken as $-\alpha C_0 u_b$, with α ranging from 1 to 10.

The conditions used at $j=0$ and $j=44$ are

$$\begin{aligned}u=v=\frac{\partial\theta}{\partial r}=\frac{\partial\sigma}{\partial r}=\frac{\partial q}{\partial r}=0\quad&\text{for}\quad j=0,\\ v=\frac{\partial u}{\partial r}=\frac{\partial\theta}{\partial r}=\frac{\partial\sigma}{\partial r}=\frac{\partial q}{\partial r}=0\quad&\text{for}\quad j=44.\end{aligned} \tag{6.4}$$

Linear interpolations were used between $j=0$ and $j=1$, in conjunction with the symmetry conditions at $j=0$, to obtain the various quantities for $j=0$.

The time integrations for the cases of group I were carried out up to thirty-six (36) hours, after which the computations have been discontinued because apparent steady states have been reached. The results of these computations are depicted in the Figs. 3a through 7. The time used for a 36-hour computation on an IBM 7094 computer is about 12 minutes.

The results of the experiment Ia are shown in the Figs. 3a-3f. It is seen that the maximum surface layer tangential velocity u_b increased gradually from its initial value of 10 m sec^{-1} to 22.7 m sec^{-1} after thirty-two (32) hours, after which it remained steady. The radius of the maximum u_b shifted gradually inward from its original position of 141.42 km to 75 km in 36 hours. On the other hand, the mean tangential velocity $\bar{u}$ given by this model changed only slightly, with its maximum increased from the original 10 m sec^{-1} to about 14 m sec^{-1} in 12 hours and thereafter stayed steady, with a slight outward shift of its position, as is seen from Fig. 3b.

Fig. 3c shows the development of the surface layer radial velocity v_b while Fig. 3d shows the development of the mid-level ω-velocity. In this computation, the mean circulation in the rz-plane becomes well established within two (2) hours after the heating effect has come into play, but not instan-

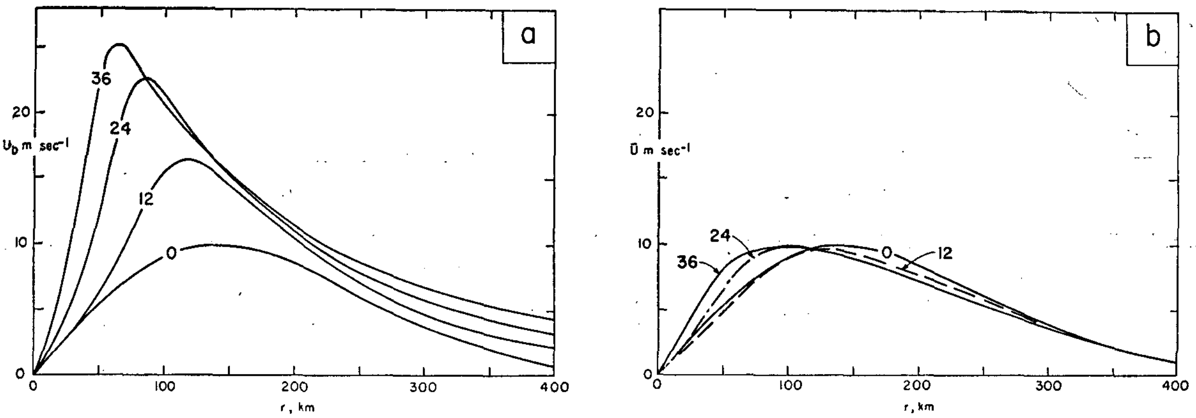

FIG. 4. Predicted variations in experiment Ib of (a) low level tangential velocity u_b (b) mid-level tangential velocity $\bar{u}$.

taneously, showing that the primary cause of the circulation obtained here is heating, not surface friction, as can also be seen from the relative contributions from these two sources to the forcing function H. From the initial time to about eight hours thereafter this circulation is composed of two cells: a main direct cell occupying the region $r>40$ km while a very weak reverse cell occupying the inner region. Descending motion in the outer main cell takes place outside $r=250$ km before $t=6$ hr, and outside $r=490$ km after $t=36$ hr, and is rather weak and uniform, while the ascending motion is relatively concentrated in the early hours and becomes widespread later on. The weak inner cell disappeared after 8 hours.

The radial inflow v_b reached its maximum intensity of about 2 m sec^{-1} in about 4 hours, after which it tapered off gradually, but the heating of the inner part of the upper atmosphere persisted and the radial temperature gradient continued to rise, as is seen from Fig. 3e, which depicts the development of the radial temperature distribution at the 250-mb level. Thus, after 36 hours a radial temperature difference of about 10C has been established. On the other hand, the radial temperature gradients at the middle and low levels are comparatively small, and cooling instead of warming has taken place in the outer parts at the 950-mb level, due largely to the excessive upward eddy transfer of heat through the upper boundary of the surface layer. Apparently, the K value used in these computations is somewhat too large.

The results of the computations for experiment Ib are represented in the Figs. 4a and 4b. The difference between this case and case Ia is in the mean-layer velocity $\bar{u}$, which did not intensify but merely shifted its maximum toward the center, as is seen from Fig. 4b. The maximum surface layer tangential velocity u_b obtained after 36 hours in this experiment is 25.4 m sec^{-1} and is therefore somewhat stronger than in Ia. It is located at about 63 km from the center. The variations of the other quantities are very similar to that of Ia and therefore they are not presented here.

Figs. 5a through 5f show the results for experiment Ic. From Fig. 5a we see that for this case the maximum u_b reached is only about 15 m sec^{-1} while the maximum $\bar{u}$ attained is about 14 m sec^{-1}. The time used in attaining these maximum values are 24 and 12 hours, respectively, after which they become steady. The development of radial and vertical winds are illustrated in Figs. 5c and 5d, which revealed that for this case these wind fields decreased rapidly toward very small values after attaining their maximum intensities at about $t=3$ hours, due to the lower surface temperature of the ascending air, as is seen from Fig. 5f. This explains the lack of further development of the tangential wind field. However, the upper level radial temperature gradient attained in this case is not very different from that in case Ia, even though that of the lower levels are weaker (see Figs. 5e and 5f). We notice that the surface layer temperature did not change beyond 200 km, on account of the suppression of the vertical eddy transfer.

The results of experiment Id are depicted in the Figs. 6a-6f. It is seen that the tangential velocity u_b obtained in this experiment is similar to that of experiment Ia except that it is slightly stronger and that the pace of development is faster; it reached a quasi-steady state after 20 hours. The meridional circulation is also more intense, with the inflow velocity v_b reaching a maximum value of 2.5 m sec^{-1} after 4 hours. The 250-mb temperature anomaly after 16 hours is about 16C, about 6C higher than in experiment Ia, showing the difference of behavior of the temperature equations (5.8) and (5.8a). The mean velocity $\bar{u}$ changed only very slightly, as is in experiment Ib.

Experiment Ie uses the same equations as in experiment Ia except that the factor α for surface friction is taken as 10. In this experiment the large friction prevented development of the surface layer tangential velocity right from the beginning, so that u_b diminishes gradually all the way and the meridional circulation remained weak.

The value $\alpha=5$ was used in experiment If. The system is then just able to maintain a maximum u_b of about 10 m sec^{-1}, while the mean-layer tangential velocity $\bar{u}$ increased from 10 to 13 m sec^{-1}. The 250-mb temperature anomaly reached after 36 hours is 8C, about 2C higher than in experiment Ie.

Two main cases have been investigated in the group II experiments, with $b=0$ or 1, and with different heating relations. As mentioned earlier, we consider the case with $b=0$ because it gives a surface-layer inflow in the whole range of a cyclonic vortex irrespective of the intensity of the surface tangential flow. The initial conditions used in these experiments are those given by (6.2), (6.2a) and an initial lapse-rate $\sigma=12.8$C.

Experiment IIa is based on $b=0$ and the heating functions

$$N_1=1.135\frac{l}{\Delta t}(\theta_s-\theta), \quad N_2=0.1N_1.$$

The results of this experiment are depicted in the Figs. 7a-7d. Both u_3 and u_2 approached their respective maximum values of 19.5 m sec^{-1} and 14.7 m sec^{-1} in about 12 hours, much faster than the corresponding rate of development in the group I experiments where no frictional inflow is imposed. In this case the radii of u_3 and u_2 maxima also shifted inward, but no equilibrium state has been established, as in the similar experiments in I. Instead, the tangential flow decayed gradually after reaching its maximum intensity. The reason for this development is the expansion of the reverse radial flow at the level ζ_3 after $t=10$ hours, as is seen from Fig. 7c, even though the surface layer inflow v_b remained inward all the time. This development seems to

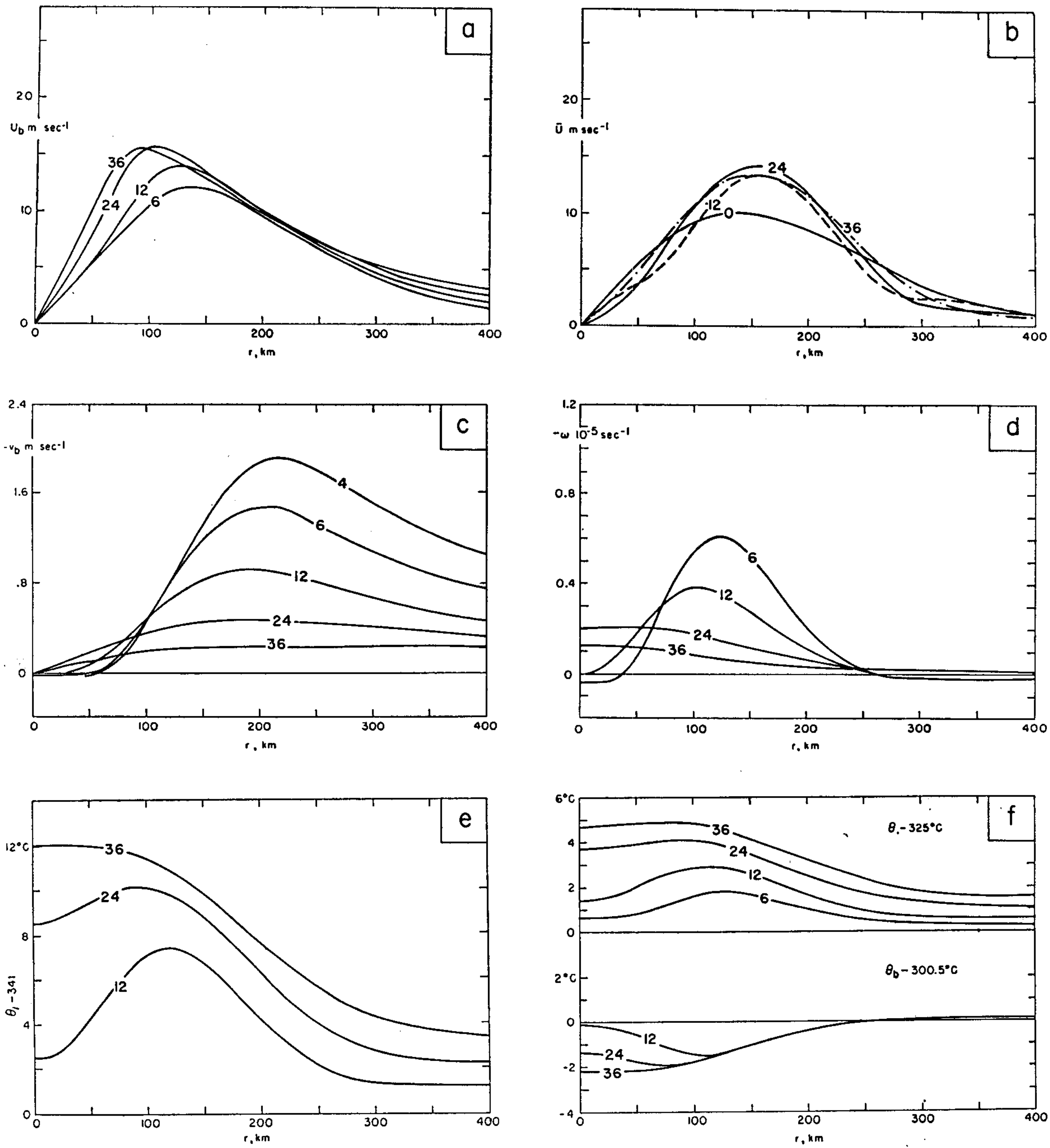

FIG. 5. Predicted time and space variations in experiment Ic of various quantities; attached numbers on curves indicate time in hours.

a) Low level tangential velocity u_b
b) Mid-level tangential velocity $\bar{u}$
c) Low level radial velocity v_b
d) Vertical ζ-velocity ω
e) Upper level potential temperature θ_1
f) Middle-level and low-level potential temperatures θ and θ_b.

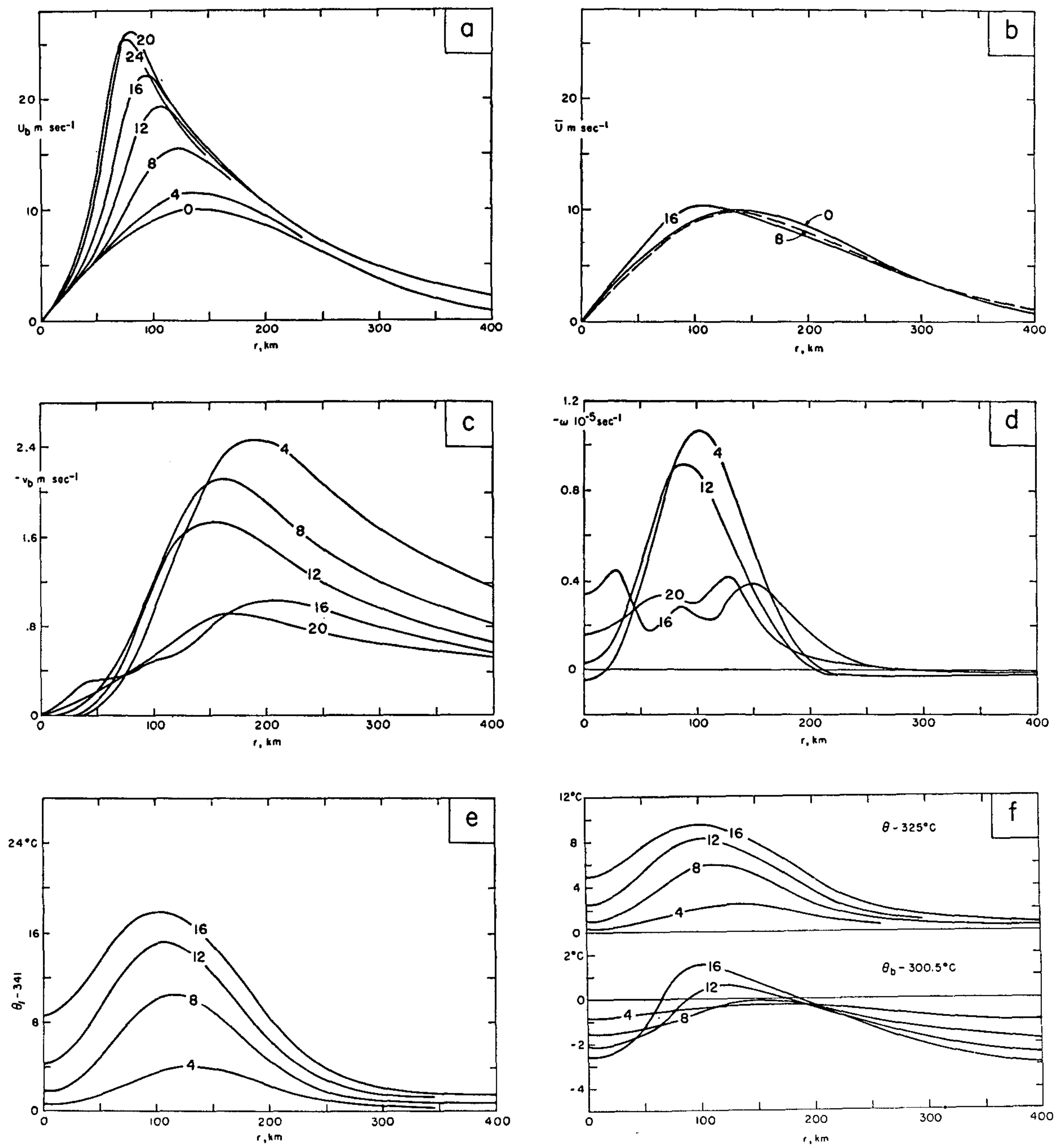

FIG. 6. Predicted time and space variations in experiment Id of various quantities; attached numbers on curves indicate time in hours.

a) Low level tangential velocity u_b
b) Mid-level tangential velocity $\bar{u}$
c) Low level radial velocity v_b
d) Vertical ζ-velocity ω
e) Upper level potential temperature θ_1
f) Middle-level and low-level potential temperatures θ and θ_b.

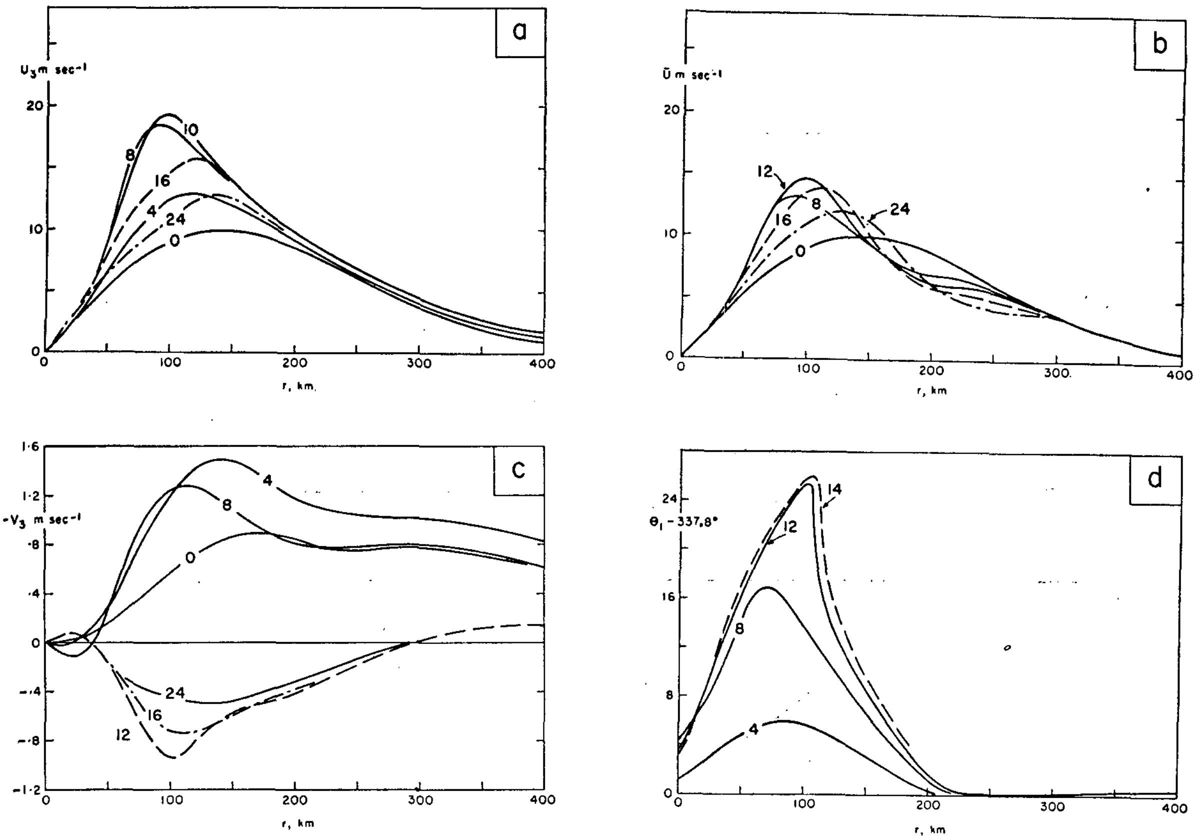

FIG. 7. Predicted time and space variations in experiment IIa of:

a) Lower level tangential velocity u_3
b) Mid-level tangential velocity $\bar{u}$
c) Lower level radial velocity v_3
d) Upper-level potential temperature θ_1.

be related to the establishment of the large negative radial temperature gradient created in the inner part of the vortex by the rapid heating in the region of maximum tangential flow, as can be seen from the distribution of θ_1 in Fig. 7d.

The case with $b=1$ and the same heating functions has also been investigated. In this case a reverse inner cell established itself right from the beginning and expanded gradually outward, so that the development of the tangential flow is reversed much earlier ($t=5$ hr) than the case with $b=0$. The maximum u_3 reached in this experiment is only 13.3 m sec^{-1}.

Experiment IIb uses the simplified heating functions (5.10c′)

$$N_1=-180\omega_4, \quad N_2=-30\omega_4 \quad \text{for} \quad \omega_4<0$$
$$N_1=N_2=0 \qquad \text{for} \quad \omega_4\geq 0$$

and therefore does not contain the self-regulating mechanism contained in the relation (5.10b). It is similar to that used by Ogura (1964), Charney and Eliassen (1964) and Ooyama (1963), except that a change of the static stability σ is included. Figs. 8a-d depict the results of this experiment for $b=0$ from $t=0$ to $t=10$ hr, after which the computation becomes unmanageable. It is seen that for this case the tangential motion intensifies with an accelerating rate, and u_3 and u_2 reached 36.7 m sec^{-1} and 22.8 m sec^{-1}, respectively, in 10 hours. The radii of the u_3 and u_2 maxima are gradually shifting inward, as is the radius of the maximum ascending motion (Fig. 8c). We note that in this experiment the maximum radial velocity v_3 remained almost constant but its position shifted inward, resulting in an increase of the maximum ascending velocity. The radial temperature gradient is also increasing continuously, as is seen from Fig. 8d.

The corresponding case with $b=1$ has also been integrated. It results in a much faster growth of the flow field, with u_3 reaching 21.5 m sec^{-1} from its original value of 10 m sec^{-1} in 4 hours and thereafter the computation becomes unmanageable.

In experiment IIc, to examine the significance of the variable stability factor σ we also used the heating relation

$$N_1=-200\omega_4 \quad \text{for} \quad \omega_4<0$$
$$=0 \qquad \text{for} \quad \omega_4\geq 0$$

with a constant $\sigma=13.7$C. The integration is performed with a uniform linear grid $r=i\Delta r, \Delta r=5$ km for the range

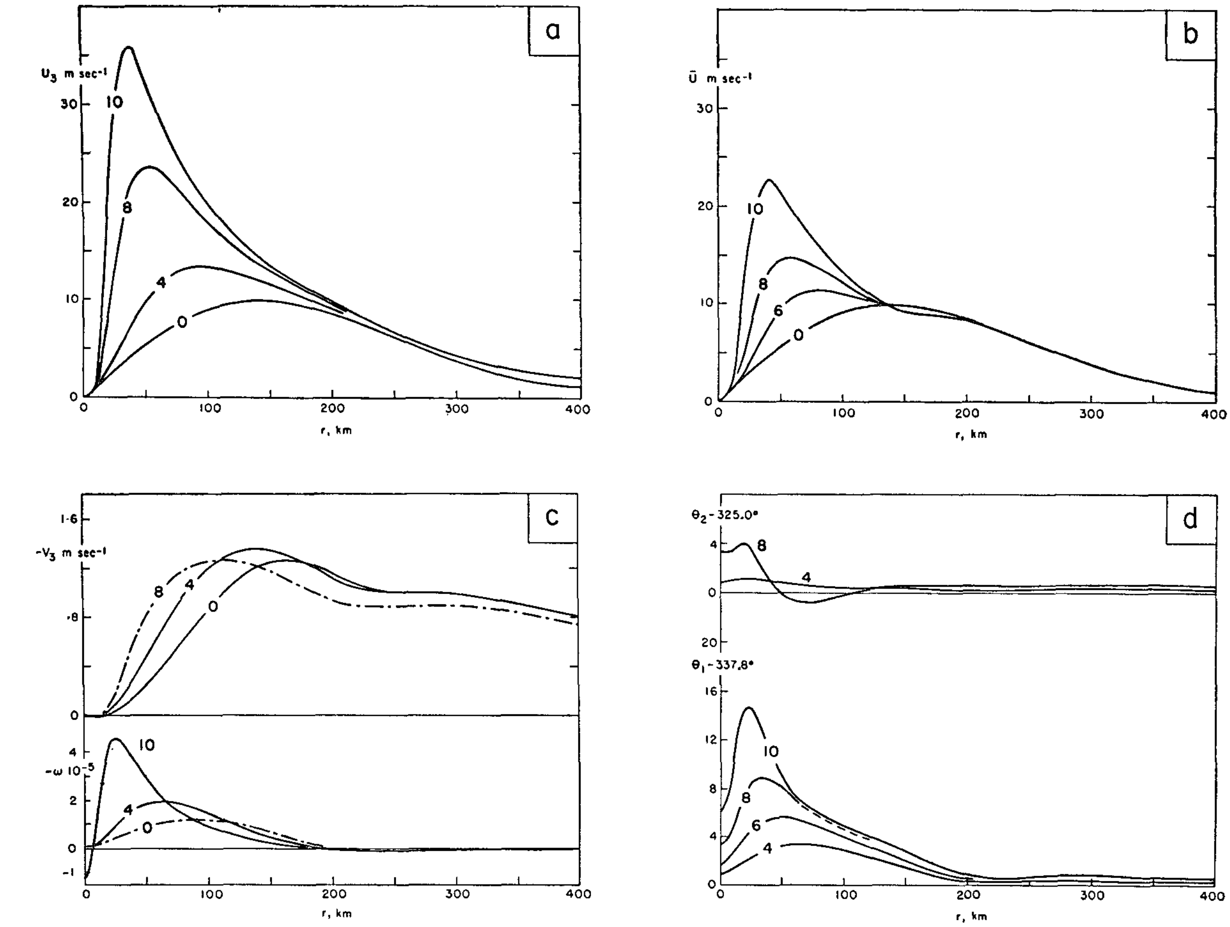

FIG. 8. Predicted time and space variations in experiment IIb of:

a) Lower level tangential velocity u_3
b) Mid-level tangential velocity $\bar{u}$
c) Lower level radial velocity v_3 (top) and Mid-level ζ-velocity ω (bottom)
d) Mid-level potential temperature θ_2 (top) and Upper-level potential temperature θ_1 (bottom).

$0 \leqslant r \leqslant 1000$ km. The initial velocity is again given by (6.2) but with a maximum velocity of only 5 m sec^{-1} at $r = 100$ km [$L = 141.42$ km). The developments of the tangential flows u_3 and u_2 for $b = 0$ are very similar to those of experiment IIb, as can be seen from Figs. 9a and 9b, except that it takes much longer to reach the unmanageable value of about 40 m sec^{-1} for u_3. The variations of the meridional velocity components v_3 and ω are illustrated in Fig. 9c, which shows that their maxima have also shifted inward, as in IIb. The temperature gradient reached at $t = 16$ hr is still insignificant, the total difference is only about 2.4C.

With $b = 1$, the development progresses at a much faster rate and it becomes unmanageable after about 7 hours, even though the maximum velocity reached then is only about 10 m sec^{-1}. Thus these last two integrations seem to indicate that allowing σ to vary is insufficient to stabilize the system when a frictional flow is included.

7. Concluding remarks

The numerical integrations show that the inclusion of the latent heat input through deep cumulus convection in the two-level balanced model without a separate friction-layer usually leads to the gradual growth of the originally barotropic system toward a steady baroclinic system with moderately strong tangential velocity, while the model with a superimposed frictional flow leads either to unlimited growth or decay. Thus the customary consideration of frictional-flow layer must be refined when applied to the hurricane intensification problem. Further, it has also been shown that the two-level balanced model is inadequate to describe the development following the establishment of a large radial temperature gradient in upper levels. Our most profitable next step of investigation of the hurricane development problem therefore appears to be the integration of a multiple-layer unbalanced model, with the latent heat input included in the manner described in Section 3.

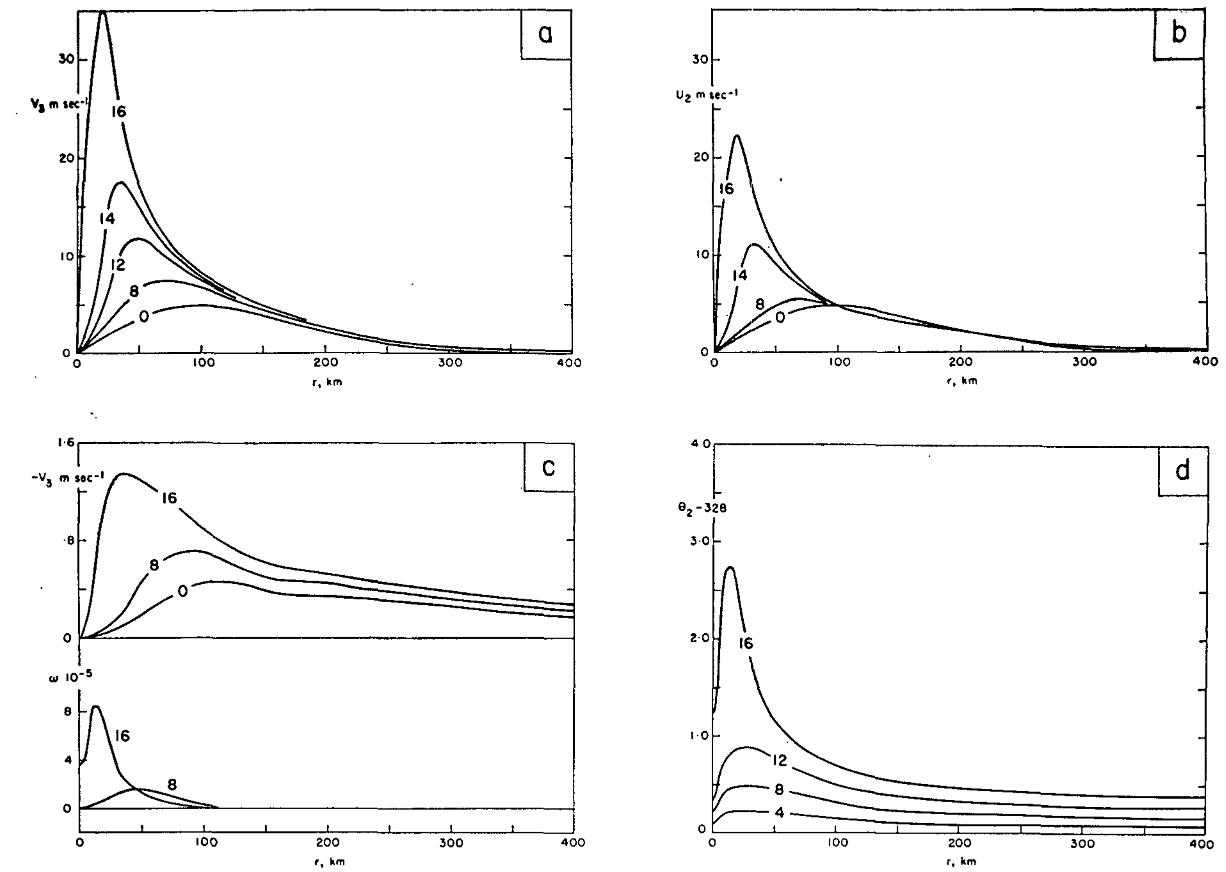

FIG. 9. Predicted time and space variations in experiment IIc of:

a) Lower level tangential velocity u_3
b) Mid-level tangential velocity $\bar{u}$
c) Lower level radial velocity v_3 (top) and Mid-level ζ-velocity ω (bottom)
d) Mid-level potential temperature θ.

APPENDIX

Formulas for Calculating T_{cb} and θ_{s2}

The formulas (5.10c) for N_1 and N_2 involve the quantity θ_{s2} explicitly through the factor $(\theta_{s2}-\theta_2)$ and implicitly through l. Therefore the calculations of N_1 and N_2 require the determination of θ_{s2}, or $T_{s2}=0.816\ \theta_{s2}$. Since T_{s2} is the ζ_2-level temperature of the moist-adiabat of the mean surface air, it can be obtained from the condensation temperature T_{cb} and the mean moist-adiabatic lapse-rate, therefore to determine T_{s2} we must at first determine the condensation temperature T_{cb}.

Since T_{cb} is the temperature the mean surface air achieves when lifted adiabatically until it becomes first saturated, T_{cb} is connected with T_b by

$$T_{cb}=T_b\left(\frac{p_c}{p_b}\right)^{R/cp}, \tag{A1}$$

where p_c is the condensation level pressure. On the other hand, since the mixing ratio q remains constant during this unsaturated lift, the change of the total pressure is related to the change of the vapor pressure through the following relation:

$$\frac{de}{e}=\frac{dp}{p}. \tag{A2}$$

Integrating this equation and substituting in the above expression for T_{cb} we obtain

$$T_{cb}=T_b\left(\frac{e_c}{e_b}\right)^{R/c_p}, \tag{A3}$$

where e_c is the vapor pressure at the condensation level ζ_{cb} and is therefore equal to the saturation vapor pressure $e_s(T_{cb})$.

Substituting $e_s(T_{cb})$ from (3.5a), we then find that T_{cb} is given by the following equation in terms of T_b and q_b:

$$T_{cb}\exp\left[2.87\left(\frac{273}{T_{cb}}-1\right)\right] = 13.54T_b^{0.396}\left[6.425\left(1+\frac{0.622}{q_b}\right)\right]^{0.1136}. \tag{A4}$$

After obtaining T_{cb} from this highly transcendental equation, we can use it in (A1) to determine p_c. However, for this two-level approximation we may use a dry-adiabatic lapse rate of 10C per 100 mb for the determination of p_c or ζ_{cb}:

$$T_{cb}=T_b-100(\zeta_b-\zeta_{cb}).$$

Projecting T_{cb} to the standard level $\zeta=1$ along the moist adiabat with a mean moist-adiabatic lapse rate of 4.1C per 100 mb for the lowest layer we obtain the surface temperature T_{s0} of the moist adiabat:

$$\begin{aligned} T_{s0}&=T_{cb}+41(\zeta_b-\zeta_{cb}) \\ &=0.41T_b+0.59T_{cb}. \end{aligned}$$

Projecting T_{s0} along the moist adiabat to the level $\zeta_2=\frac{1}{2}$ with a mean moist-adiabatic lapse rate of 4.9C per 100 mb for the lower half of the atmosphere we obtain

$$\begin{aligned} T_{s2}&=T_{sb}-24.5\text{K} \\ &=0.41T_b+0.59T_{cb}-24.5\text{K} \end{aligned} \tag{A5}$$

which has appeared in the text as Eq. (5.15). Finally, we obtain θ_{s2} from

$$\begin{aligned} \theta_{s2}&=1.225T_{s2} \\ &=0.495\theta_b+0.723T_{cb}-30.0\text{K} \end{aligned} \tag{A5a}$$

which expresses θ_{s2} in terms of θ_b and T_{cb}.

We observe that Eq. (A4) is of general validity while (A5) is limited to the two-level approximation only.

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