Further Studies of the Parameterization of the Influence of Cumulus Convection on Large-Scale Flow

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(Manuscript received 9 August 1973, in revised form 22 March 1974)

ABSTRACT

The parameterization scheme devised by the author in a previous study has been extended to include both deep cumulus convection and shallow convection and a more rigorous derivation is given. In this scheme, the amounts and the vertical distributions of the latent heat released and the sensible heat transported by the deep cumulus are expressed solely in terms of the temperature difference between the cloud and the environment and the convergence of moisture produced by the large-scale flow. It is shown that the often stressed heating by compression in the descending region is automatically taken into consideration in this formulation. A comparison between the calculated results and the observational data of Reed and Recker for the composite easterly wave show that they are in good agreement in the regions of low-level convergence.

A separate scheme is devised from the energy equations to represent the transports of heat and moisture by the shallow convection maintained by the thermal boundary layer.

1. Introduction

It is well known that many of the large-scale disturbances in the tropical atmosphere are driven by the release of latent heat and that the release of latent heat in these large-scale systems takes place mostly in deep cumulus towers and cumulonimbi. Since the horizontal scale of cumulus clouds is many orders of magnitude smaller than the grid scale used in representing large- and mesoscale flows, the influence of the small-scale convective motions can only be incorporated in the large-scale equations parametrically. The parameterization scheme devised by the author about a decade ago (Kuo, 1965) was based on a non-steady deep cumulus model, using the temperature difference between the cumulus cloud and the undisturbed environment and the large-scale convergence of moisture as indicators, while the schemes developed by Ooyama (1971), Arakawa (1971) and Yanai (1971) used the vertical mass transport as the representative variable. The relationships between these schemes and my old scheme have been discussed by Fraedrich (1973).

As the main theme of my aforementioned paper was hurricane modelling, only the essense of this parameterization scheme was described in that paper and no detailed derivation was given. As a consequence of this brevity, many misunderstandings have arisen about this parameterization scheme. One of these is the notion that it has neglected the heating of the slowly descending environment by adiabatic compression; another is on the reality of the horizontal mixing process invoked in the scheme. The purpose of this paper is to present a rigorous derivation of this parameterization scheme and to show that both the adiabatic heating of the descending branch and the adiabatic cooling of the ascending branch of the convective cell have been taken into consideration. It will also be shown that the horizontal mixing mechanism is achieved simply by the horizontal averaging of the randomly distributed convective elements over the representative area of the large-scale system, even though vigorous horizontal mixing will undoubtedly also be accomplished physically by these randomly distributed and ever-changing convection cells. In addition, I shall also extend the scheme to include the effects of the small-scale convective motion on the vertical sensible heat and momentum transfers.

It is evident that different problems demand different parameterization schemes. For simplicity, I shall classify the cumulus convections into two different categories and treat them separately; namely, 1) deep cumulus towers and cumulonimbi, and 2) shallow and mostly non-precipitating cumuli. The cumulus towers are considered as occurring only in areas of large-scale convergence and conditional instability in a deep layer of the lower troposphere, while shallow convection is considered as controlled by the heat and moisture supplies at the surface. Different formulas are derived in Sections 3 and 4 to represent the influences of convection for these two different types of convection on the large-scale flow.

2. Equations for the large-scale system and problems of parameterization

To clarify the essential points of the parameterization problem, we consider that the large-scale flow variables are represented by a horizontal grid system with grid length $\delta x=\delta y=\Delta$, and a time interval δt. The value of the large-scale flow variable χ_{ij} at any grid point can then be taken as representing the average of χ over the area $A=\Delta^2$ centered at this point, while the actual value of χ is given by the sum of the average value and the departure, *viz.*

$$\chi=\bar{\chi}+\chi', \quad \bar{\chi}=\frac{1}{A}\int_A \chi dA, \quad \bar{\chi}'=0. \tag{2.1}$$

The grid area A is taken as much larger than the area occupied by a single small-scale system such as a cumulus cloud and its surrounding descending region such that a large number of cloud cells are included in A. The exact location and time of occurrence of the individual cells are considered as unknown from the large-scale point of view, and a random distribution in space A and time δt can be assumed such that their average transport of sensible heat and release of latent heat in A and during the time interval δt are taken as determinable from the large-scale flow variables. The last point mentioned above is a necessary requirement for the feasibility of a parametric representation of the net influence of small-scale cumulus convection and other subgrid-scale processes on the large-scale system.

Thus, the equations for the potential temperature θ, the water vapor mixing ratio q, and the horizontal velocity $\mathbf{V}$ of the large-scale system can be written in (x, y, p, t) coordinates as

$$\frac{d\bar{\theta}}{dt}-Q_r-\frac{L}{C_p}\bar{C}=\frac{L\pi}{C_p}\bar{C}^*-\frac{\partial\overline{\omega'\theta'}}{\partial p}-\nabla\cdot\overline{\mathbf{V}'\theta'}, \tag{2.2}$$

$$\frac{d\bar{q}}{dt}+\bar{C}-T_q=-\bar{C}^*-\frac{\partial\overline{\omega'q'}}{\partial p}-\nabla\cdot\overline{\mathbf{V}'q'}, \tag{2.3}$$

$$\frac{d\overline{\mathbf{V}}}{dt}+f\mathbf{k}\times\overline{\mathbf{V}}+\nabla\Phi-\mathbf{F}=\frac{\overline{\omega'\partial\mathbf{V}'}}{\partial p}-\overline{\mathbf{V}'\cdot\nabla\mathbf{V}'}, \tag{2.4}$$

where $\bar{C}$ and C^* are the condensation rates produced by the large-scale motions and by the subgrid-scale convective motions, respectively, L is the latent heat of condensation, Q_r the heating rate by radiation and turulent diffusion, T_q and $\mathbf{F}$ the rates of turbulent diffusion of moisture and momentum, and d/dt the rate of change observed by following the large-scale flow, *viz.*

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla+\bar{\omega}\frac{\partial}{\partial p},$$

and $\pi=(P/p)^{R/C_p}$. The other symbols have their usual meanings. Here we have put all the contributions from the subgrid-scale flows to the right-hand side of the equations. Since the hydrostatic equation and continuity equation are not altered by the presence of the subgrid-scale flow, they need not be considered in the present consideration.

For simplicity, we shall assume that the horizontal transports by the eddies can be represented by horizontal diffusion in terms of the large-scale variables, i.e.,

$$\nabla\cdot\overline{\mathbf{V}'\chi'}=\nabla\cdot(K_\chi\nabla\bar{\chi}), \tag{2.5}$$

where K_χ is the horizontal eddy diffusion coefficient for the quantity χ. Then the last terms on the right-hand sides of Eqs. (2.2)–(2.4) can be taken as included in the Q_r, T_q and $\mathbf{F}$ terms of these equations and hence can be omitted from the present consideration.

According to the equations given above, our parameterization problem for large-scale flow is simply to strive to find expressions for the statistical contributions to the rate of latent heat release and vertical transports of sensible heat, moisture and horizontal momentum by the subscale convective motions in terms of the large-scale variables, and in doing so, there is really no need to concern ourselves with the detailed behavior of the individual convective eddy elements such as the individual cloud and its environment.

To simplify our problem, we shall assume that the statistical influence of a family of clouds of different forms and sizes can be represented by a family of model clouds and their environments, distributed randomly in the large-scale grid area A. We can then simply consider their influences in a unit area. Assuming that at any given moment the active clouds, including their strongly descending parts, occupy the fractional area a, while their environments occupy the fractional area $(1-a)$, and denoting the flow variables in the active cloud regions by letters with a subscript c and those in the surrounding regions by letters with a subscript d, we then have

$$\bar{\chi}=(1-a)\chi_d+a\chi_c, \tag{2.6}$$

where χ stands for either ω, $\mathbf{V}$, θ or q. In addition, we also have

$$\chi_c'=\chi_c-\bar{\chi}=(1-a)(\chi_c-\chi_d), \tag{2.7a}$$

$$\chi_d'=\chi_d-\bar{\chi}=-\frac{a}{(1-a)}(\chi_c-\bar{\chi}). \tag{2.7b}$$

Therefore, the average vertical eddy transport of χ can be expressed in terms of $\omega_c'\chi_c'$ alone, *viz.*

$$\overline{\omega'\chi'}=(1-a)\omega_d'\chi_d'+a\omega_c'\chi_c'=\frac{a}{1-a}\omega_c'\chi_c',$$

$$=\frac{a}{(1-a)}(\omega_c-\bar{\omega})(\chi_c-\bar{\chi}). \tag{2.8}$$

Thus, the vertical transport terms can be calculated when ω_c and χ_c have been obtained from the model of cumulus convection in terms of the large-scale flow variables.

We point out again that our ω_c represents the mean vertical velocity of the whole active cloud, including its strongly descending part, and hence it is generally much smaller than the vertical velocity of the ascending part alone, but is much larger than $\bar{\omega}$.

Eq. (2.2) indicates that the influence of the convective motions on the large-scale temperature distribution is through the release of latent heat and the mean vertical eddy transport of sensible heat, and not through heating by adiabatic compression, as is often proclaimed. Even though this last mentioned influence is important for the temperature change in the descending branch of the convection cell, it is almost cancelled by the cooling by expansion in the ascending region. With the definition of the large-scale mean value given by (2.1) or (2.6), the net effect of these two adiabatic changes is represented by the mean vertical advection term in $d\bar{\theta}/dt$. To show this more clearly, we write the heat equation for the ascending and the descending regions separately. For convenience, we shall omit the influences of the horizontal advection and the vertical eddy transfer of heat from the present consideration and use $\partial\theta_{c1}/\partial t$ and $\partial\theta_{d1}/\partial t$ to represent the temperature changes brought about by the release of latent heat and by the adiabatic expansion and adiabatic compression only. We then have

$$\frac{\partial\theta_{c1}}{\partial t}+\omega_c\frac{\partial\bar{\theta}}{\partial p}=Q_c, \tag{2.9a}$$

$$\frac{\partial\theta_{d1}}{\partial t}+\omega_d\frac{\partial\bar{\theta}}{\partial p}=0, \tag{2.9b}$$

where Q_c is the latent heat heating effect in the cumulus region. On multiplying (2.9a) by a and (2.9b) by $(1-a)$ and taking the sum, we find that

$$\frac{\partial\bar{\theta}}{\partial t}+\bar{\omega}\frac{\partial\bar{\theta}}{\partial p}=aQ_c=\frac{L}{C_p}\bar{C}^*\pi. \tag{2.10}$$

Thus, the influences of the adiabatic heating and cooling by compression and expansion in the descending and the ascending regions are automatically represented by the mean vertical advection term in this equation, and hence they need not be considered separately. In fact, if the convective velocities are defined as yielding no net upward mass transport, then those influences cancel each other exactly and will not affect the mean temperature. Thus, the influences exerted by the subscale convective motions on the mean temperature are due to the latent heat release and the vertical eddy transport of sensible heat. Notice that the latent heat released tends to heat up the atmosphere at every level in the whole region of convection, while the eddy vertical transport of sensible heat only tends to redistribute the temperature with height.

For simplicity, we shall classify the convective motions roughly into two different categories: 1) cumulus towers controlled by low-level convergence and 2) shallow cumulus convection and dry convection. The contributions of these two different types of convective motions to the large-scale flow shall be treated separately in the flowing sections.

3. Parameterization of deep cumulus convection controlled by low-level convergence

Deep cumulus towers and cumulonimbi appear to be definitely associated with a deep conditionally unstable layer and the presence of large-scale convergence (Riehl, 1950; Riehl and Malkus, 1961). The former of these two conditions makes it possible for huge cumuli to penetrate into the upper troposphere and the lower stratosphere, while the latter condition provides a general lifting mechanism to trigger the convective instability. The parameterization scheme previously devised by the author (Kuo, 1965) is specifically aimed at this kind of cumulus convection, using the difference between the temperature T_c of the active cloud core and that of the undisturbed environment, $\bar{T}$, as one representative parameter. Here I shall re-derive this formula from a somewhat simpler and more direct consideration and also take into consideration the vertical transport of sensible heat by the convective motion. The applicability of this scheme can be summarized by the following conditions:

$$D_1D_2\Delta\theta_e>C_1, \tag{3.1a}$$

$$-\tau_0\omega_b>3(p_s-p_c), \tag{3.1b}$$

where $\Delta\theta_e$ is the maximum difference of the equivalent potential temperature in the conditionally unstable layer, D_1 the depth of this layer, D_2 the height difference between the level where θ_e is minimum to the level above when θ_e is equal to its maximum below, and C_1 a critical value of the product on the left side of (3.1a), whose value can be determined empirically; τ_0 is the period of the large-scale flow, p_s-p_c the lift needed for the surface air at the level p_s to become saturated, and $\bar{\omega}_b$ the maximum low-level p-velocity, which we shall take as that at the 900-mb level. Here condition (3.1a) ensures that the instability is sufficient to support the penetrative deep convection, while condition (3.1b) ensures that this instability will be activated by the large-scale flow.

As in the author's earlier work, we shall use the net convergence of moisture into the vertical column of air of unit cross section produced by the large-scale flow and by evaporation from the ground as one fundamental parameter. Denoting this quantity by M_t, we then have

$$M_t=-\frac{1}{g}\int_0^{p_s}(\nabla\cdot\bar{\mathbf{V}}\bar{q})dp+\rho_0C_DV_0(q_g-q_0), \tag{3.2}$$

where q_g is the value of q at the surface, q_0 is that at a nearby level, and C_D is the drag coefficient.

We assume that a fraction $(1-b)$ of the total convergence of moisture M_t is condensed and either precipitated out as rain or carried away, while the remaining fraction b of M_t is stored in the air to increase the humidity, including the influence of evaporation of condensed water. That is to say, we have

$$M=(1-b)M_t\left\{\begin{array}{l}[\text{precipitated or carried}\\ \text{away part of the mois-}\\ \text{ture convergence}]\end{array}\right. \tag{3.2a}$$

$$\frac{1}{g}\int_0^{p_0}\frac{\partial q}{\partial t}dp=bM_t. \tag{3.2b}$$

We expect b to be much smaller than 1 in the regions of low-level convergence in the tropics. The vertical distribution of this part is given by $(q_c-\bar{q})$.

With the total precipitation rate M in the area as known from the large-scale convergence of q, we can now write the rate of the release of latent heat by deep cumulus convection (aQ_c) formally as

$$aQ_c=\frac{g(1-b)L}{C_p(p_b-p_t)}M_t\cdot N(p)\cdot\pi, \tag{3.3}$$

where $N(p)$ is the vertical distribution function of Q_c which satisfies the condition

$$\int_{p_t}^{p_b}N(p)dp=p_b-p_t. \tag{3.3a}$$

Now we shall express $N(p)$ and ω_c in terms of the temperature departure $\theta_c'[=\theta_c-\bar{\theta}]$ by making use of the heat equation and the notion of the life-span of the individual convection cell. Thus, in consistency with the assumption that the cross section of deep cumuli is almost independent of height so that the net vertical mass flux is negligible, we can take Q_c as given by

$$Q_c=-\frac{L}{C_p}\omega_c\left(\frac{\partial q_c}{\partial p}\right)_{\text{ef}}, \tag{3.4}$$

where $(\partial q_c/\partial p)_{\text{ef}}$ represents the effective gradient of the mixing ratio in the whole convective layer, which may be taken as equal to that of the conditionally unstable layer. Substituting this expression of Q_c in (2.9a), we then have

$$\frac{\partial\theta_{c1}}{\partial t}=-\omega_c\left(\frac{\partial\theta_e}{\partial p}\right)_{\text{ef}}, \tag{3.5}$$

where $(\partial\theta_e/\partial p)_{\text{ef}}$ is the effective lapse rate of the equivalent potential temperature.

According to observational studies (see Byers and Braham, 1949), all cumulus convection cells go through life cycles of growth and decay. The average life-span of the deep convection cells is of the order of 1 hr. Let us denote this average half-life by τ. Then the average rate of change of the temperature of the cumulus cloud from inception to maturity can be represented by

$$\frac{\partial\theta_{c1}}{\partial t}=\frac{1}{\tau}(\theta_c-\bar{\theta}), \tag{3.6}$$

where θ_{c1} is the potential temperature in the mature cloud and $\bar{\theta}$ the potential temperature of the undisturbed large-scale environment.

Equating this expression of $\partial\theta_{c1}/\partial t$ to that in (3.5), we then find

$$\omega_c=-\frac{(\theta_c-\bar{\theta})}{\tau\left(\frac{\partial\theta_e}{\partial p}\right)_{\text{ef}}}. \tag{3.7}$$

Substituting this ω_c in (3.5) we find that aQ_c in (2.9a) can be written as

$$aQ_c=K(\theta_c-\bar{\theta}), \tag{3.8}$$

where

$$K=\frac{aL}{\tau C_p}\left(\frac{\Delta q}{\Delta\theta_e}\right), \tag{3.8a}$$

where Δq and $\Delta\theta_e$ are the increments of mixing ratio and equivalent potential temperature in the conditionally unstable layer.

Now, the amount of moisture needed to create a deep cumulus of area a and pressure depth (p_b-p_t) with temperature T_c and saturation mixing ratio $q_s(T_c)$ is

$$aM_c=\frac{a}{g}\int_{p_t}^{p_b}\left[\frac{C_p}{L}(T_c-\bar{T})+q_s-\bar{q}\right]dp. \tag{3.9}$$

This must be supplied by the large-scale convergence of moisture in time τ. Hence we have

$$aM_c=\tau M_t. \tag{3.10}$$

Thus, the fractional area occupied by the deep cumulus cloud can be determined by the ratio M_t/M_c and the half-life τ of the cloud. Using this relation in (3.8a) we then find

$$K=\frac{LM_t}{C_pM_c}\frac{\Delta q}{\Delta\theta_e}. \tag{3.8a*}$$

This relation shows that the coefficient K in (3.8) is determined by the large-scale flow and the large-scale parameters and is independent of the small-scale variables. Further, since K is independent of p, Eq. (3.8) shows that Q_c is proportional to the difference between the potential temperature θ_c of the cumulus cloud and the mean value $\bar{\theta}$ at the level in question. Therefore, the vertical distribution function $N(p)$ in Eq. (3.3) is equal

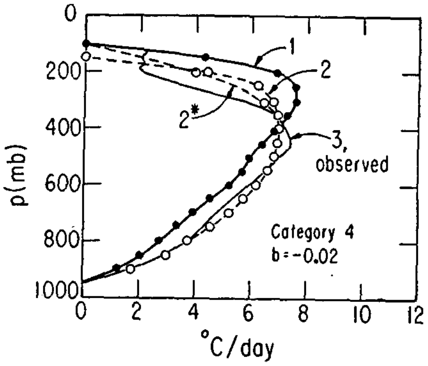

FIG. 1. Distribution of the latent heat Q_c released by deep cumulus clouds in the trough region (category 4) of the composite easterly waves: curve 1, calculated with zero entrainment; curve 2, calculated with $\gamma_1=2$; curve 2*, a slight modification of 2; curve 3, from Reed and Recker's observations.

to the ratio between $(\theta_c-\bar{\theta})$ and its mean value, *viz.*

$$aQ_c=\frac{g(1-b)LM_t}{C_p(p_b-p_t)}\frac{(\theta_c-\bar{\theta})\pi}{\langle\theta_c-\theta\rangle}$$

$$=\frac{g(1-b)LM_t}{C_p(p_b-p_t)}\frac{(T_c-\bar{T})\pi}{\langle T_c-T\rangle}, \qquad (3.11)$$

where the angle brackets represents the average over the depth of the cloud, i.e., from the cloud top $p=p_t$ to the cloud base $p=p_b$. Since $\langle\theta_c-\bar{\theta}\rangle$ can be determined more easily and more accurately than the coefficient K in (3.8a*), Eq. (3.11) is more convenient than Eq. (3.8).

The relations between this parameterization scheme of aQ_c and those given by Arakawa (1971), Ooyama (1971) and Yanai (1971) have been discussed by Fraedrich (1973).

Since this part of the parameterization is concerned with deep cumulus convection for which the influence of entrainment is relatively insignificant, the temperature T_c of the cloud can be taken as given by the temperature T_{es} of the moist adiabat through the condensation level of the representative surface layer as a first approximation. The influence of entrainment on T_c can readily be taken into consideration either by the employment of some simple empirical formula or by the use of a simple entrainment model. Thus, it is reasonable to assume that the rate of entrainment increases with ω such that the influence of entrainment on T_c can be taken as proportional to the integral of certain power of (p_b-p) multiplied by $\bar{T}$. For simplicity, we shall take $T_c-\bar{T}$ as given by

$$T_c-\bar{T}=T_s-\left[1+\sum_j^J\alpha_j\left(1-\frac{p}{p_0}\right)^{\gamma_j}\right]\bar{T}, \qquad (3.12)$$

where the summation is over the different types of tall cumulus represented by the index j, α_j is a constant determined by the vertical extent of the particular type of tall cumulus, and γ_j is another constant which is most probably greater than 2. When no entrainment is taking place, all α_j are zero. For the deep cumuli, the influence of the entrainment term is usually small except at high levels. Cumulus clouds with different base levels will not be considered because they rarely occur. The cloud top is assumed to be at the level p_t where T_c equals $\bar{T}$.

The parameterization formulas (3.11) and (3.12) have been tested with Reed and Recker's (1971) observational data for the regions 3, 4 and 5 of the composite easterly waves for which low-level convergence prevails; the results for the trough region 4 are represented in Fig. 1. Curve 1 is given by the no-entrainment version for one type of cloud only[1], and curve 2 corresponds to one type of cloud with an entrainment factor $\gamma_1=2$ and an α value corresponding to $Q=0$ at the 950- and 150-mb levels; curve 2* represents a modification of curve 2 by allowing the buoyant cloud to reach the 100-mb level. It is seen that these results are in very good agreement with the observed values below the 400-mb level. Since the observations are not very accurately determined above this level, it does not seem advisable to introduce more complicated cloud ensembles to bring the computed results into still better agreement with the observations. Calculations from the zero and nonzero entrainment versions of (3.11) and (3.12) have also been carried out for the other regions and it is found that the calculated results are in good agreement with the observations for all the convergence-dominated regions (3, 4, 5) but bear little relation with the observations for the divergence-dominated regions (1, 6, 7, 8) just as was expected.

The factor b in (3.11) is much smaller than 1 in all the regions 3, 4 and 5, just as we have predicted. A similar conclusion has also been arrived at by Betts (1973) from his observational studies over Venezuela. Thus, it appears safe to assume b to be much smaller than 1 in the regions of low-level convergence in the tropics, which implies that all the moisture convergence brought about by the large-scale flow is precipitated as rain.

Eq. (3.12) can also be used to represent the horizontal momentum departure $(\mathbf{V}_c-\bar{\mathbf{V}})$ and the specific humidity departure $(q_c-\bar{q})$ if T_c, $\bar{T}$ and T_s are replaced by $\mathbf{V}_c$, $\bar{\mathbf{V}}$ and $\mathbf{V}_s$ or by q_c, $\bar{q}$ and q_s, respectively, where $\mathbf{V}_s$ and q_s are the values of $\mathbf{V}$ and q in the surface layer.

By using the formula (3.12) for the perturbation quantity χ_c' and (3.7) for ω_c and neglecting $\bar{\omega}$ as compared to ω_c, we find that the vertical eddy transports of sensible heat, horizontal momentum and specific humid-

[1] This curve was obtained from (3.11) and the observed temperature profile by Dr. H. R. Cho. I would like to express my gratitude to him for allowing me use of this curve here.

ity can all be calculated by the equation:

$$\overline{\omega'\chi'}=-\frac{M_t}{(1-a)M_c}\frac{(\theta_c-\bar{\theta})}{\left(\frac{\partial\theta}{\partial p}\right)_{\rm ef}}(\chi_c-\bar{\chi}), \qquad (3.13)$$

where χ represents either θ, q or $\mathbf{V}$.

According to this formula and Eqs. (3.11) and (3.9), the net vertical flux of sensible heat produced by the deep cumulus convection is given by

$$-\frac{\partial\overline{\omega'\theta'}}{\partial p}=\frac{M_t}{(1-a)M_c}\left(\frac{\partial\theta}{\partial p}\right)^{-1}_{\rm ef}\frac{\partial}{\partial p}(\theta_c-\bar{\theta})^2, \qquad (3.14a)$$

$$=\frac{2aQ_cC_p\langle\theta_c-\bar{\theta}\rangle}{(1-a)(1-b)\langle[C_p(T_c-\bar{T})+L(q_s-\bar{q})]\rangle}\times\left(\frac{\partial\theta}{\partial p}\right)^{-1}_{\rm ef}\frac{\partial}{\partial p}(\theta_c-\bar{\theta}). \qquad (3.14b)$$

Since $(\theta_c-\bar{\theta})^2$ has its maximum at the mid-level of the cloud layer (p_b-p_t), Eq. (3.14a) indicates that the influence of this sensible heat transport is to warm the upper portion and to cool the lower portion of this layer. But since $C_p\langle T_c-\bar{T}\rangle$ is much smaller than $L\langle q_s-\bar{q}\rangle$ and $\partial(\theta_c-\bar{\theta})/\partial p$ is smaller than $(\partial\theta/\partial p)_{\rm ef}$, this term is much smaller than aQ_c. Therefore, the thermal influence of deep cumulus convection is given mainly by aQ_c in (3.11), while the sensible heat transport is only to shift the maximum heating rate to a slightly higher altitude.

4. Parameterization of the dry and shallow cumulus convection

Observations show that in the layer from the surface to 10 or 100 m the lapse rate is superadiabatic over the land during the day when the sky is not overcast, and over the warm oceans both during the day and during the night. Above this relatively thin superadiabatic surface layer is the well-mixed layer with a lapse rate very close to the adiabatic, while above this mixed layer atmospheric stability increases rapidly with height (occasionally to the point where a temperature inversion occurs), and the humidity drops sharply. The depth of this mixed layer changes prominently during the day over the land, reaching its maximum of from 1000 to 2000 m around 1300 local time (Kuo, 1968; Vul'fson, 1961), whereas over the tropical oceans it changes only slightly and is of the order of 500 m (Malkus, 1958; Donelan and Miyake, 1973). When no synoptic-scale low-level convergence is present, only shallow cumulus occur above this mixed layer.

It is well known that vigorous convection creates complete mixing and a near-adiabatic lapse rate above the thin superadiabatic and warm surface layer (Kuo,

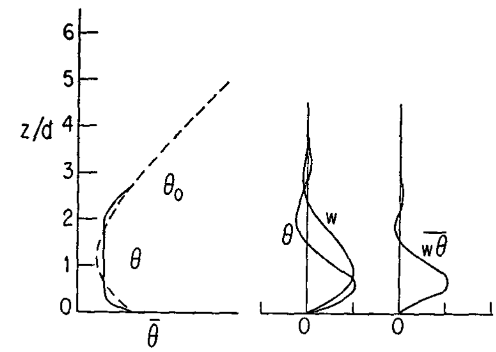

FIG. 2. Linear solutions of cellular convection equations for a superadiabatic surface layer topped by a deep stable layer. Left, schematic illustration of the influence of the vigorous convection on the mean potential temperature; middle, variations of the amplitudes of w and θ and the heat flux $\overline{w\theta}$ with height.

1961; Deardorff and Willis, 1967); hence, the existence of the mixed layer can be attributed directly to the convection. The large stability that exists on top of the mixed layer can also be explained since convection from the hot surface tends to heat the middle portion and cool the top and the bottom portions of the convective layer, as is illustrated by the solutions of the linearized convection equations in Fig. 2.

It seems evident that most of the shallow cumulus are simply manifestations of the ascending convection currents sustained by the warm surface that have penetrated above the condensation level. These circulations can be visualized as well-organized convection cells even though the ascending currents may be composed of small plumes or bubbles. The observation that some of the ascending currents enter the cloud base from the surroundings rather than directly from below can be explained by the fact that the circulations that produce the shallow cumuli are better organized convection cells of larger horizontal dimension as compared to those in the mixed layer. The uniformity of the height of the cloud base is a proof that all the ascending currents originate from the surface layer. Since the lower troposphere is usually only conditionally unstable and is unsaturated, most of the descending currents will experience the influence of the stable stratification and hence must be weaker and broader than the ascending currents, especially at upper levels.

The depth of the cumulus cloud produced by the convection cell is determined by the temperature excess T' and the vertical velocity w of the ascending current above the condensation level. If T' becomes negative and w vanishes as a result of the stable stratification above the mixed layer, then only a shallow cumulus can be produced, whereas if both T' and w are positive above the condensation level, then the potential energy of the conditional instability is made available and the convection develops spontaneously and extends to the level of the major inversion. These two different situations are illustrated schematically in Figs. 3a and 3b. Thus,

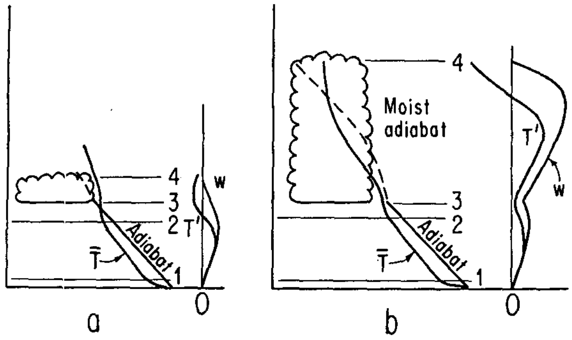

FIG. 3. Schematic presentation of shallow cumulus convection, with little penetration in the stable layer, a., and deeper cumulus convection, with conditional instability activated at upper levels, b.

according to the analysis presented above, the dry and shallow cumulus convection over the tropical oceans can be taken as of the same nature as those over the land during a clear day, the only difference being that moisture content plays a larger role and the diurnal variation is less important over the oceans.

We shall now consider the parametric representation of the influences of dry and the shallow convection on the large-scale flow. Assuming that these types of convection are controlled by the surface heat and moisture fluxes and are unrelated to the large-scale flow, we can then take the fluxes of sensible heat and latent heat across the lower surface as the given quantities for the parameterization problem. Thus, on expressing these fluxes either in terms of the temperature and humidity gradients or in terms of the temperature and moisture differences in accordance with the turbulence theory, we have

$$\Delta H=\rho K_T\frac{\partial T}{\partial z}+\frac{L}{c_p}K_q\frac{\partial q}{\partial z}$$
$$=\rho C_D V\left[(T_s-T_0)+\frac{L}{c_p}(q_s-q_0)\right], \qquad (4.1)$$

where K_T and K_q are the eddy diffusion coefficients of heat and moisture at the surface level, C_D the surface drag coefficient, V a representative wind speed and T_s, T_0, q_s and q_0 are the temperatures and mixing ratios at the surface and of the air above. We may use either one of these two expressions to represent the surface flux. The thermodynamic aspect of the parameterization problem is to express the vertical distribution of this heat input in terms of the large-scale flow fields.

One way of parametrically expressing the vertical transports of various quantities by shallow convection is simply to represent them by a diffusion formula with an eddy diffusion coefficient depending on the stability, the height, and the velocity distribution, such as that proposed by the author (Kuo, 1968) and by Deardorff (1972). However, such simple treatments are usually unable to represent convective transports accurately.

Another way is to make use of the well-established results on thermal convection in a highly unstable layer; namely, the layer will be well-mixed by the turbulent convection such that all the conservative properties for inviscid and adiabatic flow will be independent of height in the convective layer. We shall now devise a parameterization scheme for shallow convection by the application of this principle.

Since the nature of the convection in the layer above is different from that in the well-mixed layer, we shall treat the parameterization problems in these two layers separately and connect them by the partition factor β of the total surface heat flux.

a. The well-mixed layer: $z_b\leqslant z\leqslant z_m,\ p_m\leqslant p\leqslant p_b$

Since no condensation is taking place in this layer, both the horizontal momentum, the mixing ratio, and the potential virtual temperature θ_v, defined by

$$\theta_v=T_v(P/p)^{R/C_p}$$

where $T_v=(1+0.608q)T$ is the virtual temperature, will remain nearly constant in this well-mixed layer. To express the temperature changes in this layer in terms of ΔH, we assume that a fraction β of the heat ΔH supplied at the boundary during time Δt is used to compensate the radiational cooling and other heat losses in this layer. The heat balance equation for this layer can then be written as

$$\beta\Delta H=\frac{C_p}{g}\int_{p_m}^{p_b}(p/P)^{R/C_p}(\theta_{vf}-\theta_{vi})dp$$
$$=\frac{C_p}{g}\int_{p_m}^{p_b}(T_{vf}-T_{vi})dp, \qquad (4.2)$$

where θ_{vi} and θ_{vf} are the potential virtual temperatures before and after the addition of heat, p_m and p_b are the pressures at the top and the bottom of this mixed layer, and T_{vf} and T_{vi} are the corresponding virtual temperatures. Here θ_{vf} can be taken as constant throughout this layer. Thus, if the depth z_m-z_b (or p_b-p_m) and the fraction β are known, θ_{vf} can be obtained easily for any heating rate ΔH. We now assume that the heat flux convergence at level z in this well-mixed layer is proportional to $T_{vf}-T_{vi}$. Then the vertical distribution of the sensible and latent heat transported by convection can be calculated by a formula similar to (3.11), i.e.,

$$\delta Q=\beta\Delta H\frac{(T_{vf}-T_{vi})}{(\bar{T}_{vf}-\bar{T}_{vi})}, \qquad (4.2a)$$

where the overbar denotes an average over the depth of the mixed layer. In case T_{vi} is not obtainable from direct observations, it must then be determined theoretically, for example, from the solution of the combined radiation and turbulent conduction equation.

b. *The organized convection above the well-mixed layer*

As has been mentioned above, the convection in this layer can be visualized as in the form of more or less steady convection cells with their more concentrated ascending branches originating partly from the surface layer and partly from the surroundings above the mixed layer. Even though no satisfactory solution of the nonlinear equations for this case has yet been obtained, we can make use of the existing solutions of the simplified and linearized equations such as that represented in Fig. 2 and the required surface heat transport ΔH to determine the various vertical eddy transports.

Another method of calculating the vertical transports by the convection currents is to make use of a simple plume-type model, by assuming that the pressure is hydrostatic and the environment remains undisturbed. Even though such a model has many obvious shortcomings, it has the advantage of providing a clearer picture of the ascending current. Thus, we write the vertical momentum and heat equations for steady axisymmetric flow as

$$\frac{\partial uwr}{\partial r}+\frac{\partial w^2 r}{\partial z}=rs-\mu r^2 w^2, \tag{4.3}$$

$$\frac{\partial usr}{\partial r}+\frac{\partial wsr}{\partial z}=-gs_z rw-\mu r^2 sw, \tag{4.4}$$

where $s=g(\theta-\theta_0)/\theta_0$ and $s_z=\partial \ln\theta_0/\partial z$. Here a nonlinear drag proportional to w^2r^2 is added to (4.1) and a nonlinear diffusion proportional to wsr^2 is added to (4.2) to simulate part of the neglected influences of the environment on the ascending current. Further, we assume that w and s have the form

$$w=w_c(z)f(\eta),\quad s=s_c(z)f(\eta),\quad \eta=r^2/R^2(z), \tag{4.5}$$

where w_c and s_c are the values of w and s at $r=0$, $R(z)$ is the radius of the ascending current, and

$$f(\eta)=\begin{cases}(1-\eta)^{\frac{1}{2}}, & \text{for } 0\leqslant\eta\leqslant 1\\ 0, & \text{for } \quad \eta>1\end{cases} \tag{4.5a}$$

Integrating (4.3) and (4.4) over r from 0 to R and then multiplying the result from (4.3) by Rw_c, we then obtain

$$\frac{dy}{d\zeta}+\frac{4}{5}\mu y=2H, \tag{4.6}$$

$$\frac{dH}{d\zeta}+\frac{8}{15}\mu H=-\frac{4}{3}gs_z y^{\frac{1}{3}}, \tag{4.7}$$

where the variables y, H and ζ are defined by

$$y=w_c^3R^3,\quad H=w_cs_cR^2,\quad \zeta=\int_{z_b}^{z} Rdz,$$

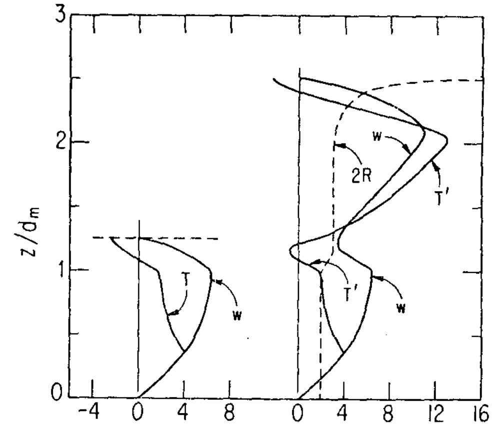

FIG. 4. Variations of the temperature perturbation and vertical velocity with height given by the plume model. Left, shallow cumulus convection; right, deeper cumulus convection when the conditional instability is activated at upper levels. The dashed curve is the variation of the plume diameter with height.

and z_b is the height of the superadiabatic surface layer. The boundary conditions are

$$y=y_0,\quad H=H_0 \quad \text{at } \zeta=0, \tag{4.8}$$

where H_0 can be obtained from ΔH in (4.1); and y_0 obtained from the following similarity solution valid for the superadiabatic surface layer with $s_z=s_{z0}z^{-4/3}$, and zero or linear friction:

$$w_c=(2gs_{z0})^{1/2}z^{1/3},\quad s_c=2gs_{z0}z^{-1/3}. \tag{4.9}$$

We assume that the area of the ascending current remains independent of height in the unstable and the neutral layers but increases with height in the stable layer according to the following mass conservation equation with a mass entrainment factor α:

$$\frac{dR^2w_c}{dz}=\alpha Rw_c,\quad \text{for } s_z>0. \tag{4.10}$$

For convenience, we write this equation as

$$\frac{dR^2}{d\zeta}+\left(\frac{2}{3Y}\frac{dY}{d\zeta}\right)R^2=\alpha, \tag{4.10a}$$

and integrate it together with (4.6) and (4.7).

The results obtained from these equations show clearly that either when the degree of instability of the surface layer is weak or when the distance between the top of the mixed layer and the condensation level is large, the ascending current will then not be able to penetrate far above the condensation level, and therefore only a thin cloud can be produced. This is illustrated by the results in Fig. 4a. On the other hand, even a slight increase of the degree of instability or a reduction of the depth of the intermediate stable layer may allow the ascending current to gain positive buoyancy above the condensation level, and thereby develop further spontaneously and create a deeper cumulus cloud. This case is illustrated by the results in Fig. 4b. Note that here the temperature and vertical velocity distributions in the cloud are similar to each other and also similar to that given in Section 3 for the deep cumulus convection controlled by large-scale convergence; the difference between this case and the large-scale controlled case is that here the convections above and below the cloud base appear to be separated from each other by the intermediate stable layer.

5. Concluding remarks

It has been shown that both the amount and the vertical distribution of the latent heat released by convergence-controlled deep cumulus convection can be calculated by the formula (3.11) in terms of the moisture convergence M_t and the temperature difference $(T_c-\bar{T})$ between the cloud and the environment, while $(T_c-\bar{T})$ itself can be obtained from the relation (3.12), which also holds for the horizontal momentum and moisture differences. In addition, the convective vertical velocity ω_c itself can be obtained from (3.7), which can then be used to calculate the vertical transports of sensible heat, horizontal momentum, moisture and other quantities by the deep cumuli. When applied to the convergence portion of the composite wave disturbances in the equatorial western Pacific, it is found that the amount and the vertical distribution of the latent heat released by deep cumulus clouds as given by these simple formulas agree well with those obtained by Reed and Recker (1971) from observations.

On the other hand, the vertical transport of heat and moisture produced by shallow cumulus unrelated to the large-scale convergence are attributed to convection from the warm surface below the cloud base and calculated in terms of the heat and moisture fluxes at the surface by the use of a plume-type model. Here it is found that the thickness and the strength of the stable layer just above the convectively mixed surface layer are the most important factors in determining whether only a thin or a thicker cumulus will develop under a given surface condition.

Finally, we point out that a time lag is probably involved between the occurrence of the moisture convergence M_t and the absorption of the latent heat $\pi LC^*/c_p = aQ_c$ by the large-scale flow, as described by Eq. (2.2), because time is needed both for the deep cumulus to develop and for the large-scale flow to adjust to the unevenly distributed Q_c.

Another point worth mentioning is that just above the cloud base, $\bar{\omega}$ is usually larger than $|\omega_d|$ and hence a thin layer of cloud will cover the whole mean ascending region. The latent heat released in this layer can be calculated from $\bar{\omega}$ and $\bar{q}$. The convergence of moisture used by $\bar{C}$ must then be subtracted from M_t in (3.10) and (3.11).

Acknowledgments. This research is supported by the National Science Foundation under Grant GA-25161-A1 and by the National Oceanic and Atmospheric Administration under Grant N22-154-72(G).

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